CAAM641 Spring 2007

The Eikonal equation $\|\nabla\Phi(x)\| = n(x)$: models, theory, simulation.

Chapter 0 : http://www.tangentspace.net/cz/archives/2006/11/eikonal-equa

- Where :

- When :

- Homework - Grading ?

Outline :

Where does it come from ? What is it used for ?

- Formal asymptotic solutions of Helmholtz
- Eikonal and Ray tracing
- Going behind caustics (Maslov theory)
- Wigner Transform
- GTD
- Gaussian Beams

- Other examples of occurence of the Eikonal equation

What do we know about ?

- Law of refraction : History - Fermat principle
- Euler Lagrange and ray tracing (CV)
- Lagrangian $\longrightarrow$ Eulerian : Classical solutions
- Basics on Viscosity solutions
- Optimal control interpretation of the viscosity solution
- Explicit formula Lax-Oleinik

- in the $(Max, +)$ Algebra ...

- Relationship with Hyperbolic Conservation Laws ...

How do we compute it ?

- Quick recap on classic methods for ray tracing
- The need for more : wavefront construction
- Viscosity solutions : derivation of a simple 1-D Upwind scheme
- Convergence theory
- Rouy-Tourin scheme in 2-D
- Fast algorithms (Fast marching, Sweeping ...)

- "LAgrangian Schemes : Explicit formula Lax-Oleinik
- Fast Legendre Transfoms
- Algorithm in $(Max, +)$ Algebra ..

Helmholtz equation

(*see also W. Symes seminar notes on time domain ... and* `www.cscamm.umd.edu/programs/hfw05/runborg_survey_hfw05.pdf`)

$$\Delta u(X) + k^2 n^2(X) u(X) = 0$$

(local) .... to make it well posed add

- a (possibly infinite) domain.
- Radiation conditions or absorbing boundary conditions .
- Source terms or boundary conditions (diffraction problem) .

Formal asymptotic solution

Debye expansion (1911) - Chap 1. Equations with Rapidly Oscillating solutions Partial Differential equations V , M.V. Fedoryuk

$$u_k(X) = e^{ik\Phi(X)} \sum_{j=0}^{\infty} \frac{1}{(ik)^j} A_j(X)$$

Apply H. operator and order in powers of k

$$\begin{aligned}\Delta u_k + k^2 n^2 u_k = \; & -k^2(\|\nabla\Phi\|^2 - n^2) + \\ & ik(2\nabla\Phi \cdot \nabla A_0 + A_0 \Delta\Phi) + \\ & (\Delta A_0 + ...) + \\ & \ldots\end{aligned}$$

FAS :

$$\{\frac{\Delta}{k^2} + n^2\}(A_0(X) e^{ik\Phi(X)}) = O(k^{-1})$$

The Eikonal equation (EE) - Ray Tracing Solution (RTS)

$$\|\nabla\Phi(X)\|^2 = n(X)^2$$

Method of characteristics (assuming $\nabla\Phi$ is smooth) :
$\frac{d}{ds}Y(s) = \nabla\Phi(Y(s))$. Set $P = \nabla\Phi(Y)$ then

$$\frac{dY}{ds} = P(s),\ \frac{dP}{ds} = \frac{1}{2}\nabla n^2(Y(s))$$

The phase Φ, can be computed as the integral of $\|P\|^2$ along a ray $Y(s)$, since

$$\frac{d}{ds}\Phi(Y(s)) = \frac{dY}{ds}\cdot\nabla\Phi(Y(s)) = \|P(s)\|^2 = n^2(Y(s))$$

Rem. 1 : All this is completely local ! BC IC ?
Rem. 2 : parameterization of paths can be changed (ex : $dt = n^2(Y(s))ds$ is a possibility)

A word about Amplitudes

Setting $B(s) = A(Y(s))$ gives (TE) :

$$\frac{d}{ds}B(s) = \frac{dY}{ds} \cdot \nabla A(Y(s)) = B(s)\ \Delta\Phi(Y(s))$$

... Instead remark that TE :

$$\nabla \cdot (A^2 \nabla\Phi) = 0$$

Integrate on a "ray tube" $\{Y(s, y_0) | \, 0 < s < t \, | \, y_0 \in B(\overline{y}_0, \epsilon)\}$ and use the divergence th. yields

$$B^2 nJ \mid_{Y(s,\overline{y}_0)} = constant$$

where $J = det(\frac{\partial Y(s,y_0)}{\partial y_0})$

Ex. 1 : $n = 1$ plane wave solution

$$u(X) = Ae^{\pm ik\xi\cdot X}$$

if $\|\xi\| = 1$.

RTS :

$$Y(s, Y_0) = Y_0 + s\xi,\ P(s, Y_0) = \xi,\ \Phi(Y(s, Y_0)) = \Phi_0(Y_0) + s$$

ICs on $z = 0$: $Y_0 = (x, 0)$, $P(0, Y_0) = \xi$, $\Phi_0(x, z = 0) = x\xi_x$ give

$$\Phi(X) = \xi \cdot X$$

Physical Optics

For a diffracion pbm ($u = Ce^{ike_\theta \cdot x} + u_{scat}$) us the integral formulation ($u = 0$ on the scatterer)

$$u(x) = \int_{\partial\Omega} G(k|x - x'|)\frac{\partial u}{\partial n}dx'$$

and the geometric optic approximation $u = Ce^{ike_\theta \cdot x} - Ce^{ike_{-\theta} \cdot x}$ to replace by

$$u(x) = \int_{\partial\Omega_{light}} G(k|x - x'|)(2Ciksin\theta)dx'$$

Ex. 2 : $Hu = \delta_{x_0}$ fundamental solution : Hankel function, point source

$$u(X) = H_0^1(k|X - X_0|)$$

As $k \to +\infty$ $u(X) \simeq (\sqrt{\frac{2}{\pi k|X-X_0|}} e^{-i\frac{\pi}{4}}) e^{ik|X-X_0|}$

RTS :

$$Y(s,\theta) = X_0 + s\vec{e}_\theta,\ P(s,\theta) = \vec{e}_\theta,\ \Phi(Y(s,\theta)) = \Phi_0(\theta) + s$$

ICs for $\theta \in [0, 2\pi[$: $Y(0,\theta) = X_0,\ P(0,\theta) = \vec{e}_\theta,\ \Phi_0(\theta) = 0$ give

$$\Phi(X) = |X - X_0|$$

Asymptotic interpretation of the R.B.C.

The solution behave asymptotically (in kr) as

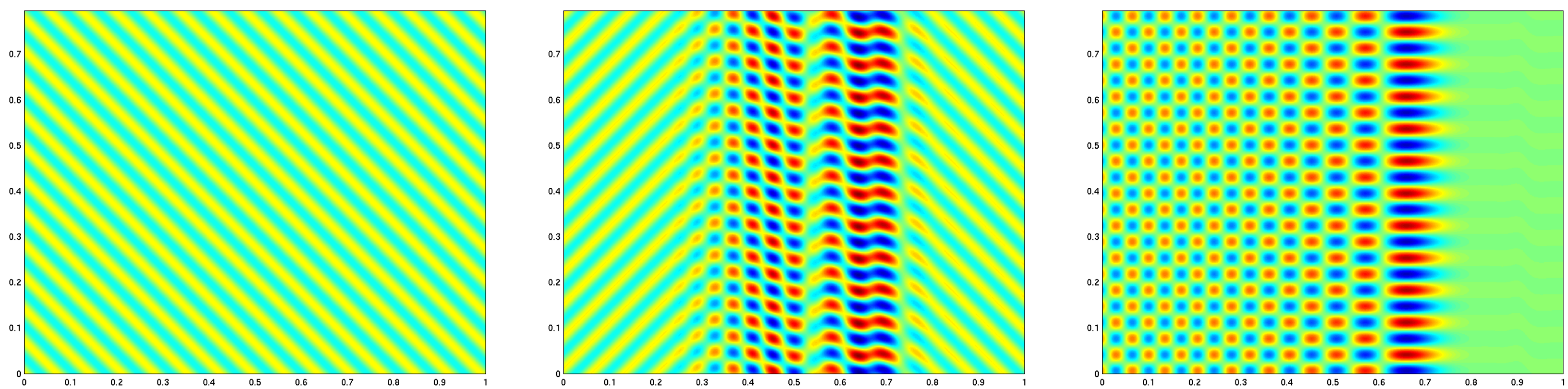
$$u \simeq A(\theta)\frac{e^{ikr}}{\sqrt{r}}$$

which is to say that far form source or scaterrer solution behaves like geometric point source in homegenous space with amplitudes modulated by θ .

Ex. 3 : A fold caustic case

$$\begin{array}{lll} gauche & x < 0 & n^2(x,z) = 1 \\ droite & x \in [0, 1-\epsilon] & n^2(x,z) = 1 - x \ , \ x \in [0, 1-\epsilon] \\ & x > 1 - \epsilon & n^2(x,z) = \epsilon \end{array}$$

$$u_{inc} = e^{ik\vec{e}_\alpha \cdot X}$$



Separation of variables $\rightarrow$ 1D

We can set $u(z,x) = \tilde{u}(x)e^{ikz\sin\alpha}$ then

$$\tilde{u}_g''(x) + k^2(\cos^2\alpha)\tilde{u}_g(x) = 0 \qquad \tilde{u}_g(x) = e^{ik\cos\alpha x} + Re^{-ikx\cos\alpha}$$

$$\tilde{u}_d''(x) + k^2(\cos^2\alpha - x)\tilde{u}_d(x) = 0 \qquad \tilde{u}_d(x) = CAi(-k^{\frac{2}{3}}\rho(x)) + DBi(-k^{\frac{2}{3}}\rho(x))$$

$$\rho(x) = \cos^2\alpha - x$$

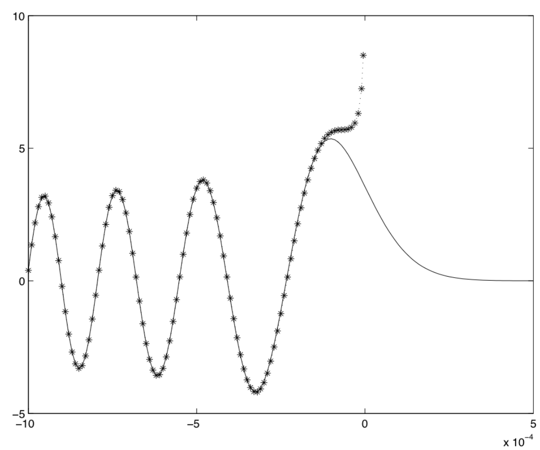
As $k \rightarrow +\infty$: On the shadow side ($\rho(x) < 0$)

$$Ai(-k^{\frac{2}{3}}\rho(x)) \rightarrow 0 \;\; Bi(-k^{\frac{2}{3}}\rho(x)) \rightarrow +\infty$$

So we take $D = 0$, determine R, C with the compatibility conditions at $x = 0$ and on the lighted side ($\rho(x) > 0$)

$$\tilde{u}_d = \frac{C}{2\sqrt{\pi}\rho^{\frac{1}{4}}}\{e^{-i\frac{\pi}{4}}e^{ik(\frac{2}{3}\rho^{\frac{3}{2}})} + e^{i\frac{\pi}{4}}e^{-ik(\frac{2}{3}\rho^{\frac{3}{2}})}\} + O(k^{-1})$$

(Stationnary phase on $Ai(-k^{\frac{2}{3}}\rho(x)) = \int e^{ik(\rho(x)\xi - \frac{\xi^3}{3})}d\xi$)



Next RTS

Only work in $x \in [0, 1-\epsilon]$, $n^2(x) = 1-x$, take $Y(s,z_0) = (y(s,z_0), z(s,z_0))$ and $P = (p_y, p_z)$ we get :

$$\begin{cases} \dot{y} = p_y & y(0,z_0) = 0 \\ \dot{z} = p_z & z(0,z_0) = z_0 \\ \dot{p_y} = -\frac{1}{2} & p_y(0,z_0) = \cos\alpha \\ \dot{p_z} = 0 & p_z(0,z_0) = \sin\alpha \\ \dot{\Phi} = 1-y & \Phi(0,z_0) = \sin\alpha z_0 \end{cases}$$

which yields $Y(s,z_0) = (-\frac{1}{4}s^2 + s\cos\alpha, s\sin\alpha + z_0)$

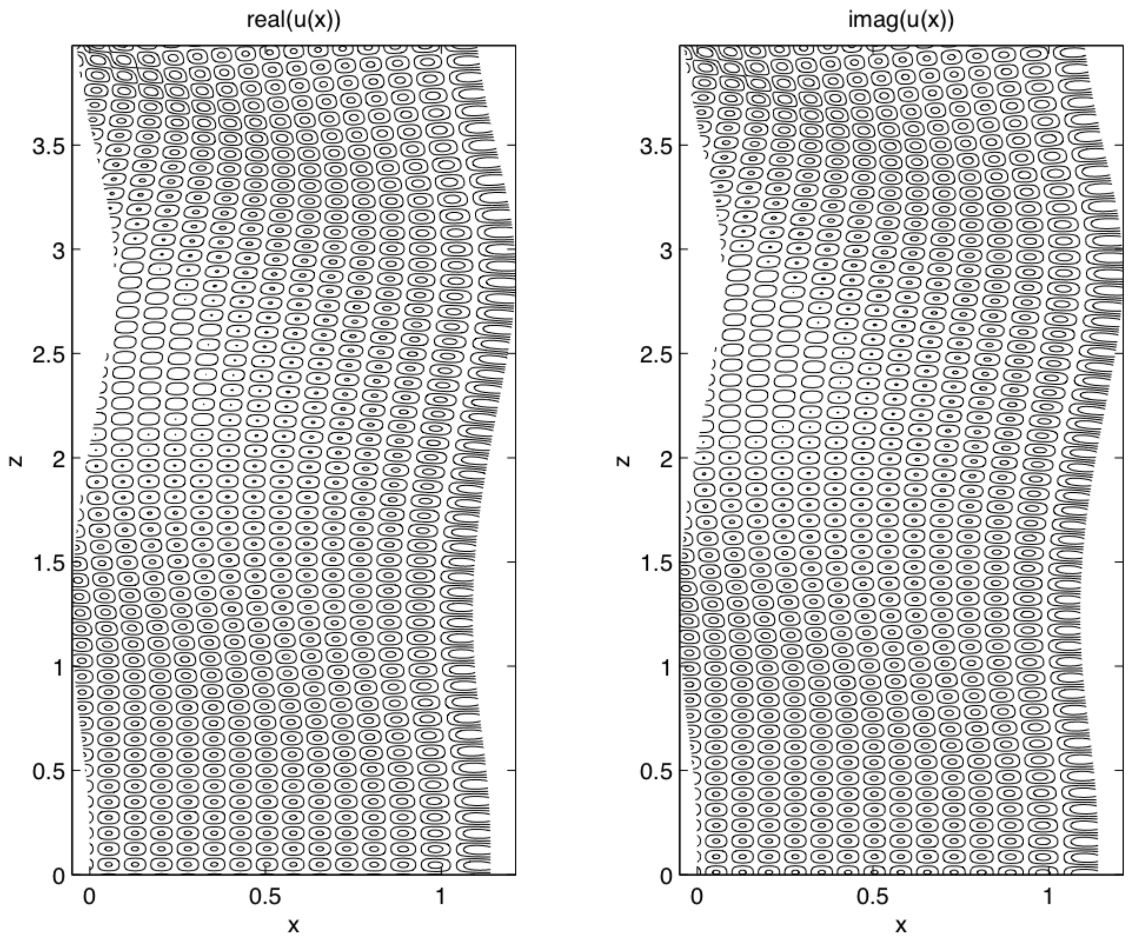
Use z as parameterization and finally

$$\text{At } (x,z) = Y(s, z_0^{\pm}) \rightarrow \text{ 2 phases :}$$

$$\Phi^{\pm}(x,z) = z \sin\alpha \pm \tfrac{2}{3}\rho(x)^{3/2} + \tfrac{2}{3}\cos^2\alpha$$

Away from a Fold caustic, stationary phase theorem gives

$$u = a^- e^{-i\pi/4}\, e^{ik\Phi^-} + a^+ e^{i\pi/4} e^{ik\Phi^+} + O(k^{-1})$$


real(u(x))
imag(u(x))
z
x
0
0.5
1
1.5
2
2.5
3
3.5

Global Asymptotic solutions (Maslov)

Need to know the "catastrophe" which ruins the standard FAS. Simplest case is the "Fold caustic" :

Consider (locally) the smooth manifold

$$M = \{(y(z, z_0), z, p_y(z, z_0), p_z(z, z_0)) \,|\, z_0 \in Z_0,\ z \in Z\}$$

then "TH." (Hörmander, Duistermaat, ...) : $\exists \phi(X, \theta)$, $X = (x, z)$ $\theta \in \mathcal{R}$ such that

1. $M = \{(X, \nabla_X \phi) \,|\, \partial_\theta \phi = 0,\ z \in Z\}$.

2. $\forall X \in M$, $\exists \theta^{\pm}(X)$ such that $\partial_\theta \phi(X, \theta^{\pm}(X)) = 0$ and

$\partial_{\theta^2}\phi(X,\theta^{\pm}(X)) \neq 0$.

3. Use generalized ansatz $u(X) = (\frac{k}{2\pi})^{\frac{d}{2}} \int A(X,\theta,k) e^{ik\phi(X,\theta)} d\theta$ and expansion $A(x,\theta,k) = \sum_{j=0}^{\infty} \frac{1}{(ik)^j} A_j(X,\theta)$ then away from Caustic, stationnary phase Th. gives

$$u(X) = \sum_{\pm} \frac{A_0(X,\theta^{\pm}(X))}{|\partial_\theta \phi(X,\theta^{\pm}(X))|^{\frac{1}{2}}} e^{\mp i\pi/4} e^{ik\phi(X,\theta^{\pm}(X))} + O(k^{-1})$$

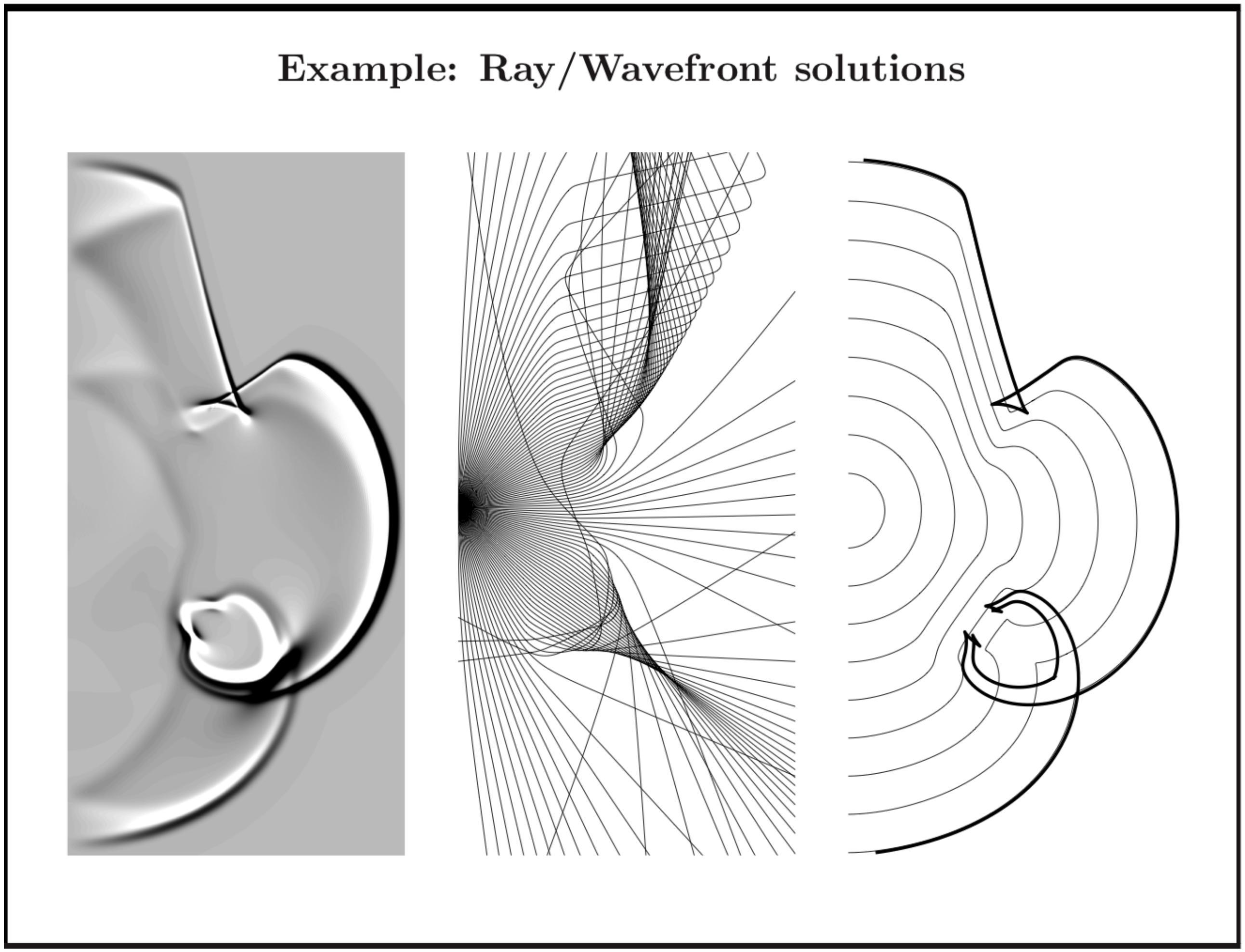
$\rightarrow$ superposition of simple AS. Computable by RTS (or in this case with 2 Eikonal Equations).

4. $\partial_{\theta^2}\phi(X,\theta^{\pm}(X)) = 0$ indicate the caustic where the stationnary point is degenerate and give a contirbution in $O(k)$.

Rem. 1 : Other option is find a change of variable such that $\phi(X,\xi) = \phi_0(X) + \rho(X)\xi - \frac{\xi^3}{3} \rightarrow$ Airy function.

Rem. 2 : Other catastrophe in 2D is the cusp which leads to Pearcey and 3 branches.

other Ex. (from Runborg)



Motivation 1 : Discretisation depend on k

Finite Element Solution of the Helmholtz Equation with High Wave Number Part II: The h-p Version of the FEM
Frank Ihlenburg; Ivo Babuska SIAM Journal on Numerical Analysis, Vol. 34, No. 1. (Feb., 1997), pp. 315-358.
For degree-p Lagrangian FE in 1D

$$\|u - u_h\| \leq C_1(hk)^p + C_2 k(hk)^{p+1}$$

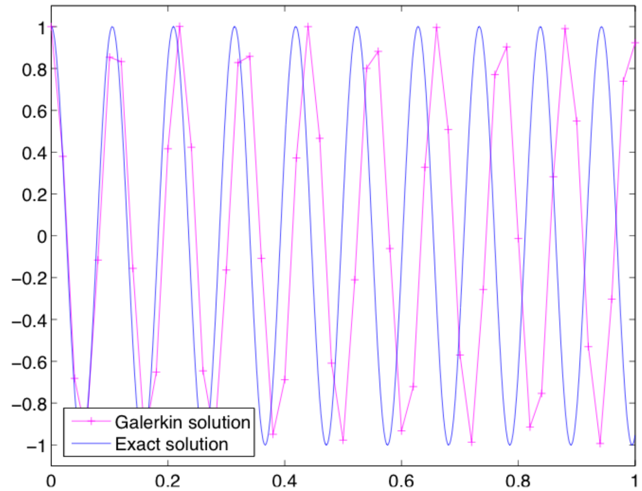
Rule of thumbs : $hk < 1 \rightarrow$ Very Large systems to solve.

`http://alphard.ethz.ch/hafner/Workshop/Hiptmair.pdf`

Numerical Dispersion

Helmholtz equation: model for propagation of time-harmonic waves

$$\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^d \quad , \quad u = 0 \quad \text{on } \partial\Omega \,.$$



($d = 1$, $\Omega =]0,1[$, $f \equiv 0$, $u(x) = \exp(i\kappa x)$, $\kappa = 40$)

← Galerkin solution, p.w. linear FE on equidistant mesh

- Phase error of Galerkin solution

  = Numerical dispersion

- Main source of discretization error in numerical wave propagation (at medium & high frequencies)

Motivation 2 : Theory and practice of Migration Operators *Mathematical Theory For Seismic Migration and spatial resolution. G. Beylkin*
check W. Symes seminar notes

`http://www.trip.caam.rice.edu/txt/tripinfo/other_list.html`

The Wigner Transform Approach

$$v_\epsilon(x,y) = u(x + \tfrac{\epsilon}{2}y)\overline{u}(x - \tfrac{\epsilon}{2}y), \;\; (x,y) \in \mathcal{R}^2$$

$$f_\epsilon(x,\xi) = \mathcal{F}_{y\to\xi} v_\epsilon(x,y) = \int e^{-iy\cdot\xi} v_\epsilon(x,y)dy$$

Some properties

$$\|u(x)\|^2 = v_\epsilon(x,0) = \{\mathcal{F}^{-1}f\}(x,0) = \int f(x,\xi)e^{i0\cdot\xi}d\xi$$

$$\epsilon Re\{u(x)\nabla\overline{u}(x)\} = \tfrac{\partial}{\partial y}v_\epsilon(x,0) = \tfrac{\partial}{\partial y}\{\mathcal{F}^{-1}f\}(x,0) = \int i\xi f(x,\xi)e^{i0\cdot\xi}d\xi$$

Looking at the Wigner transform of Assymptotic Solutions gives some insight :

$$u = A(x)\ e^{ik\phi(x)} \rightarrow v_\epsilon(x,y) = A^2(x)e^{ik(\epsilon y \nabla\phi(x) + O(\epsilon^3))} + O(\epsilon^2)$$

Taking $\epsilon = k^{-1}$ and neglecting high order terms we get :

$$f(x,\xi) = A^2(x) \int e^{iy\cdot(\nabla\phi(x)-\xi)}dy = A^2(x)\delta(\xi - \nabla\phi(x))$$

Rem. 1 : Same with sums of AS

The kinetic equation for

$\Delta u(x) + \epsilon^{-2} n^2(x) u(x) = \delta(x)$

Notice : $\nabla_y \cdot \nabla_x v_\epsilon = \frac{\epsilon}{2}[\Delta u(x+\frac{\epsilon}{2}y)\bar{u}(x-\frac{\epsilon}{2}y) - u(x+\frac{\epsilon}{2}y)\Delta\bar{u}(x-\frac{\epsilon}{2}y)]$

and thus we have :

$i\nabla_y \cdot \nabla_x v_\epsilon(x,y) + \frac{i}{2\epsilon}[n^2(x+\frac{\epsilon}{2}y) - n^2(x-\frac{\epsilon}{2}y)]v_\epsilon(x,y) = \sigma_\epsilon(x,y)$

Fourier Transform and

$$\xi \cdot \nabla_x f_\epsilon + Z_\epsilon(x,\xi) \star_\xi f_\epsilon = Q_\epsilon(x,\xi)$$

As $\epsilon \to 0$

$$Z_\epsilon(x,\xi) = \frac{i}{2\epsilon}\mathcal{F}_{y\to\xi}[n^2(x - \frac{\epsilon}{2}y) - n^2(x - \frac{\epsilon}{2}y)]$$

So formaly (Taylor expand the $n2$ functions above around x) :

$$Z_\epsilon(x,\xi) \to_{\epsilon\to 0} \frac{1}{2}\nabla_x n^2(x)\cdot\nabla_\xi\delta(\xi)$$

Can also check that

$$Q_\epsilon(x,\xi) = \mathcal{F}_{y\to\xi}[\sigma_\epsilon(x,y)] \to_{\epsilon\to 0} \delta(x)\delta(|\xi| = 1)$$

Finally :

$$\xi \cdot f(x,\xi) + \frac{1}{2}\nabla_x n^2(x) \cdot \nabla_\xi f(x,\xi) = \delta(x - x_0)\delta(|\xi| = 1)$$

which can be reformulated as a time dependent problem $f(x,\xi) = \int_0^\infty \tilde{f}(s,x,\xi)\,ds$ (particles go at infty with time) :

$$\partial_s \tilde{f}(s,x,\xi) + \xi \cdot \nabla_x \tilde{f}(s,x,\xi) + \tfrac{1}{2}\nabla_x n^2(x) \cdot \nabla_\xi \tilde{f}(s,x,\xi) = 0,$$

$$\tilde{f}(0,x,\xi) = \delta(x - x_0)\delta(|\xi| = 1)$$

Back to Ray tracing : Set

$f_0(s,x,\xi) = \sum_{\theta \in \Theta} \delta(x - Y(s,\theta))\delta(\xi - P(s,\theta))$

where $\{Y(s,\theta), P(s,\theta)\}$ satisfy the RT equations

$$\frac{dY}{ds} = P(s),\ \frac{dP}{ds} = \frac{1}{2}\nabla n^2(Y(s))$$

with point source initialisation :

$$Y(0,\theta) = x_0\ ,\ P(0,\theta) = (\cos\theta, \sin\theta)$$

satify the limit kinetic equation. Use a test function $g(x,\xi)$ and setting $< f, g >= \int fg\,dx\,d\xi$. Then one can check formally that (Y, P depend on s, θ)

$$\begin{aligned} < \partial_s f_0, g >= &\ \textstyle\sum_{\theta\in\Theta}\{\frac{dY}{ds}\cdot\nabla_x g(Y,P) + \frac{dP}{ds}\cdot\nabla_\xi g(Y,P)\} \\ = &\ \textstyle\sum_{\theta\in\Theta}\{P\cdot\nabla_x g(Y,P) + \frac{1}{2}\nabla n^2(Y)\cdot\nabla_\xi g(Y,P)\} \\ = &\ < \xi\cdot\nabla_x g(x,\xi) + \tfrac{1}{2}\nabla n^2(x)\cdot\nabla_\xi g(x,\xi), f_0 > \\ = &\ - < \xi\cdot\nabla_x f_0, g > - < \tfrac{1}{2}\nabla n^2(x)\cdot\nabla_\xi f_0, g > \end{aligned}$$

Rem . : Morrey Campanato semi-norms involved in Helmholtz case (see `B. Perthame et al, High Freqency limit of the Helmholtz equation. Rev. Iber. Americana ...`)

Rem. : As $\epsilon \to 0$ convergence is weak in $\mathcal{S}'$

`Wigner Functions versus WKB-Methods in Multivalued Geometrical Optics, Christof Sparber ? , Peter A. Markowich. y. , Norbert J. Mauser`

More : Level sets, Image processing PDE (second order terms ...)

www.intlpress.com/CMS/issue4/levelset_imaging_chapter.pdf

www.math.uci.edu/~zhao/publication/publication.html

More : Shape From Shading

perception.inrialpes.fr/Publications/2006/PF06a/chapter-prados-faugeras.pdf

More : Geodesics Mesh Refinement, Creeping rays, RT in Anisotropic media

www.nada.kth.se/~olofr/Publications/article-creep2-1.pdf

math.unice.fr/~rascle/psfiles/hr01.ps

Symes Quian elastic waves (See TRIP WWW)

More vaguely related but more Fashionable : BlackScholes (finance) 2nd order terms...

Distance functions

$d(x, x_0) = |x - x_0|$ satisfies $|\nabla_x d| = 1$.

Anticipating : $\phi(x) = \min_{x_0 \in \partial\Omega} d(x, x_0)$ is the viscosity solution of

$|\nabla\phi| = 1 \; in \; \Omega, \; \phi = 0 \; on \; \partial\Omega.$

Level sets image processing

$\frac{d\Gamma(t,\alpha)}{dt} = v(\Gamma(t,\alpha))$ and set $\Gamma(t,\alpha) = \{x, \phi(x,t) = 0\}$ then ϕ is solution of

$$\frac{\partial \phi}{\partial t} + v \cdot \nabla \phi = 0$$

Ex. 1 : Normal motion $v = n\frac{\nabla\phi}{|\nabla\phi|}$. Note n may depend on x.

Ex. 2 : Motion by mean curvature *(Evans and Spruck ....)* : $v = \nabla.(\frac{\nabla\phi}{|\nabla\phi|})\frac{\nabla\phi}{|\nabla\phi|}$

Ex. 3 : Surface reconstruction : $v = \nabla d$, $|\nabla d| = 1$, $d(x) = 0 \; for \; x \in \mathcal{S}$

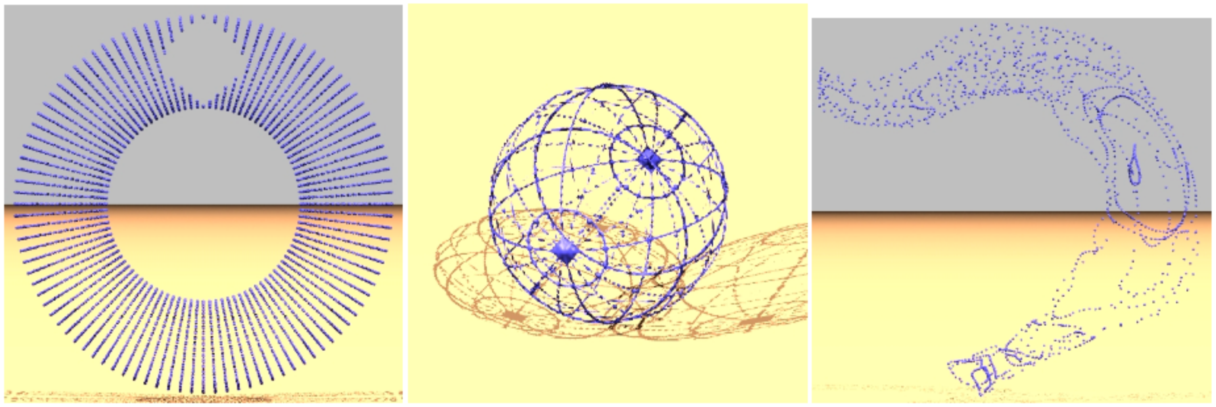
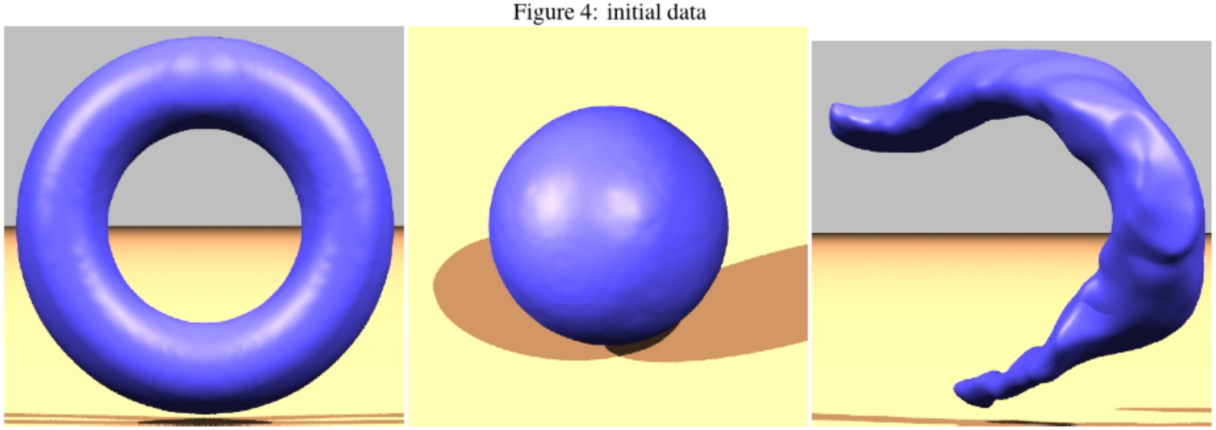
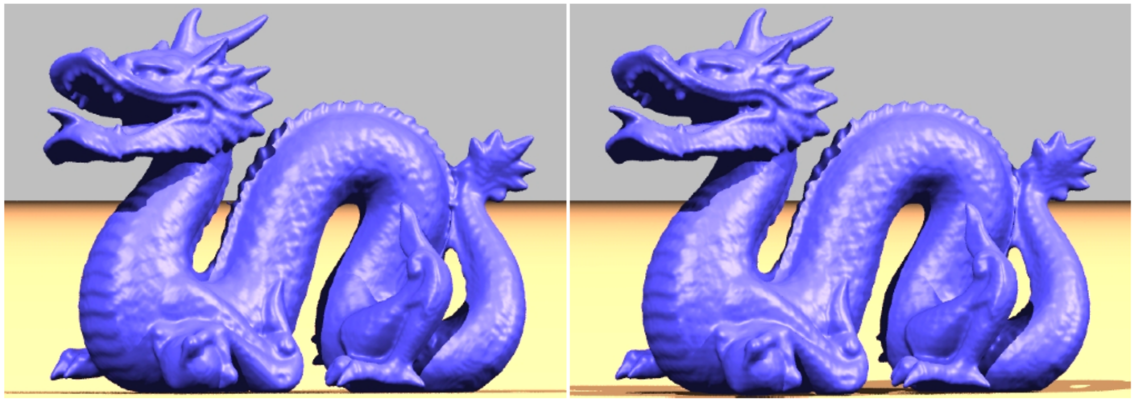


Figure 4: initial data

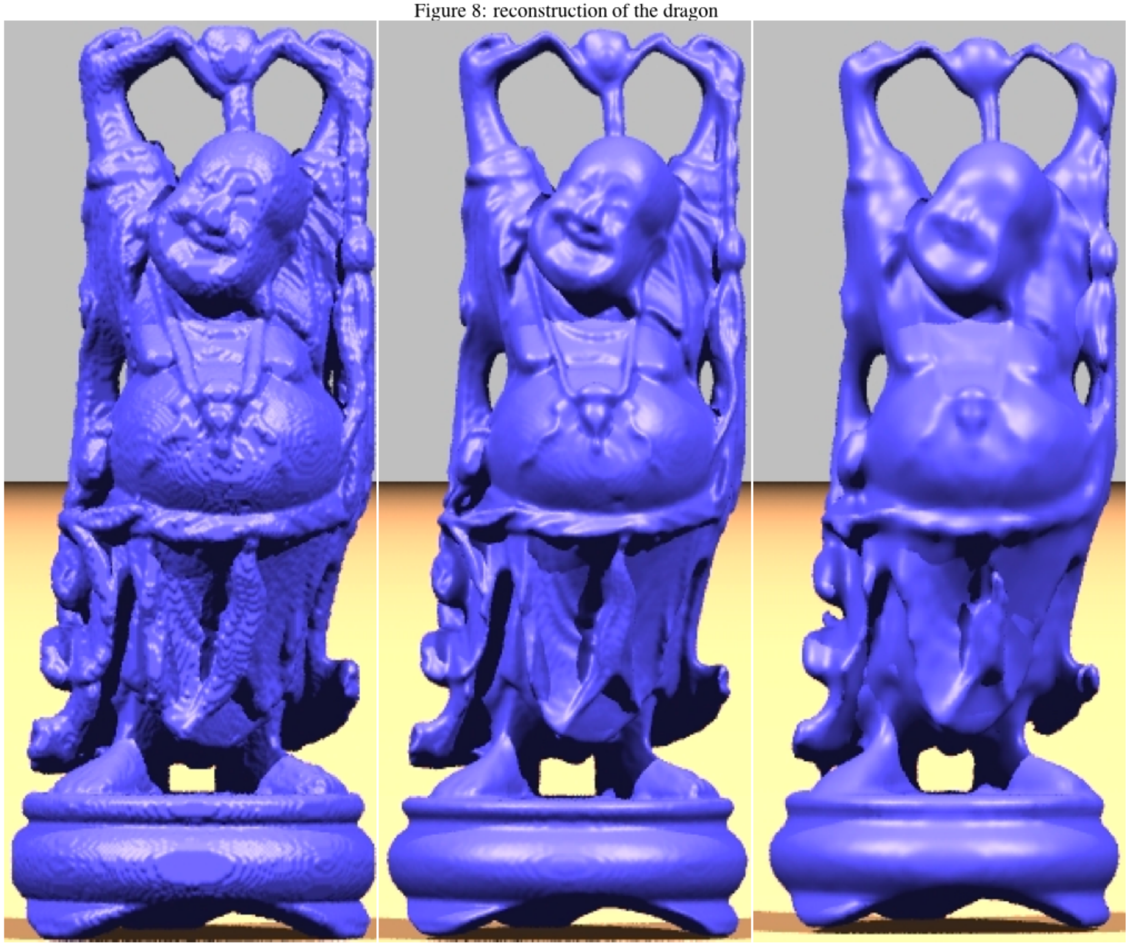


hole filling of a torus | reconstruction of a sphere from curves | reconstruction of a rat brain from MRI slices

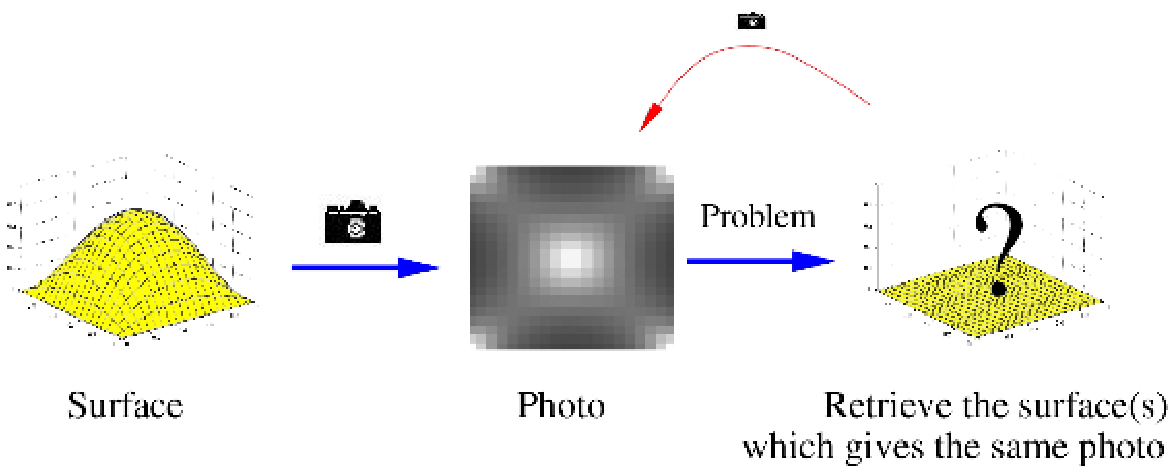


(c) final reconstruction (d) low resolution reconstruction

Figure 8: reconstruction of the dragon



Shape From Shading



Lambertian scene hypothesis (L, light and n, normal to the surface vectors) $I(x_1, x_2) = R(n(x_1, x_2)) = \cos(L, n) = \frac{L}{|L|} \cdot \frac{n}{|n|}$

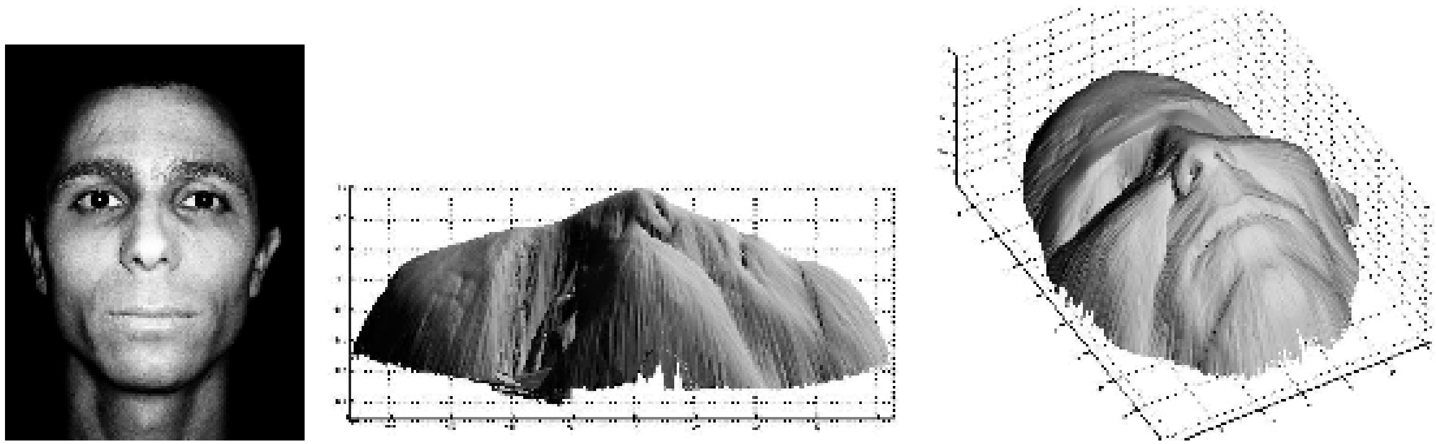
"Orthographic SFS" with a far light source

$L = (\vec{l}, \gamma)$ is constant and such that $|L| = 1$

$S : x = (x_1, x_2) \to (x, u(x))$ is a the surface parameterization so that $n(x) = (-\nabla u(x), 1)$.

$$I(x) = \frac{-\nabla u(x) \cdot \vec{l} + \gamma}{\sqrt{1 + |\nabla u(x)|^2}}$$

With $L = (0, 0, 1) \to |\nabla u(x)|^2 = \frac{1}{I^2(x)} - 1$



a) b) c)

Mesh refinements

Now use a Riemannian metric G. Note : length of dx is defined as $|dx|_{G(x)} = |G^{\frac{1}{2}} dx|$. If $G = diag(lx, ly)$ and $lx \neq ly \rightarrow$ anisotropy.

Again : $\phi(x) = \min_{x_0 \in \partial\Omega} d_G(x, x_0)$ is the viscosity solution of $|G^{\frac{1}{2}} \nabla \phi| = 1 \; in \; \Omega, \; \phi = 0 \; on \partial\Omega$.

Now define G using a surface $(x, z(x))$: $G(x) = Id + \nabla z \otimes \nabla z$ or $Id + H(z)^t H(z)$.

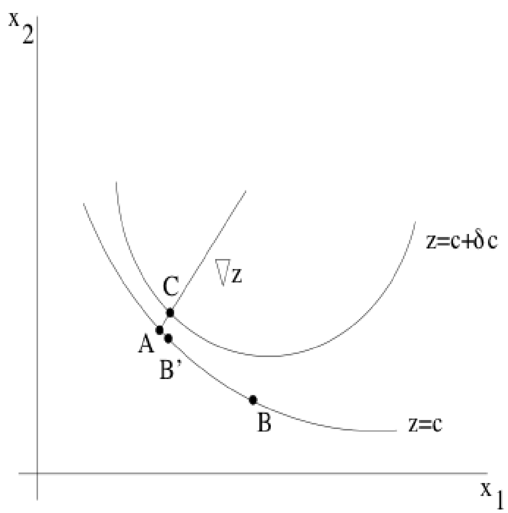


Fig 1

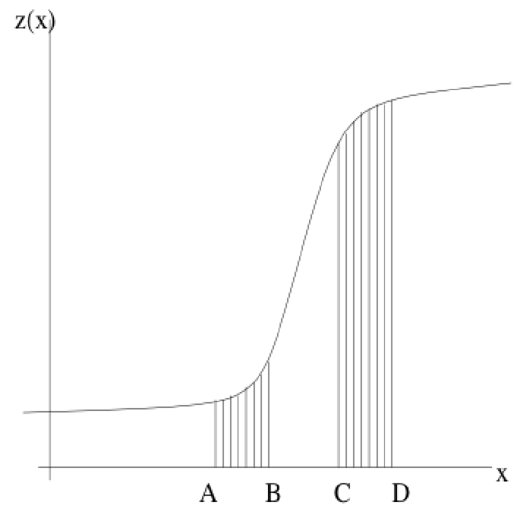


Fig 2

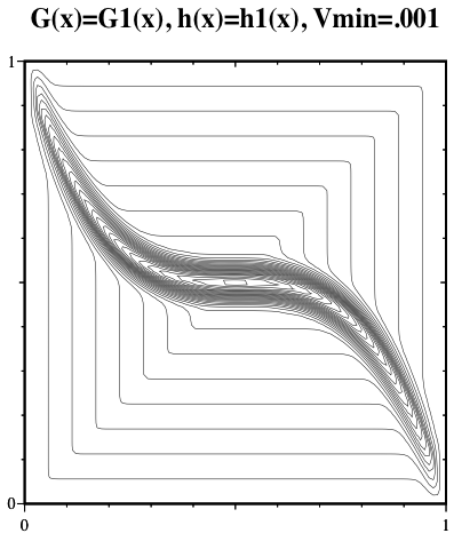


Figure 4a

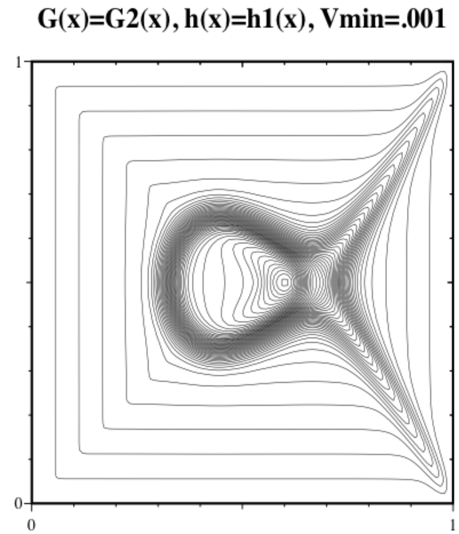


Figure 4b

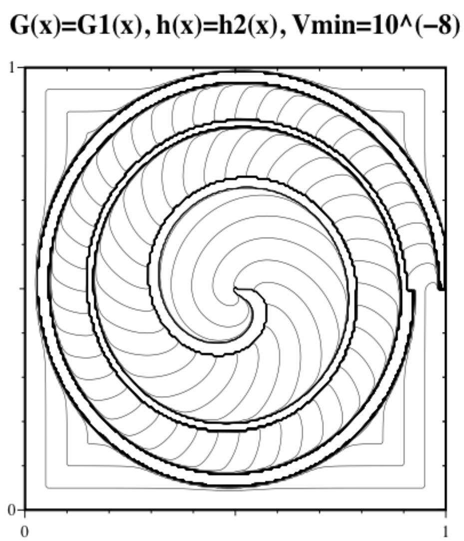


Figure 4c

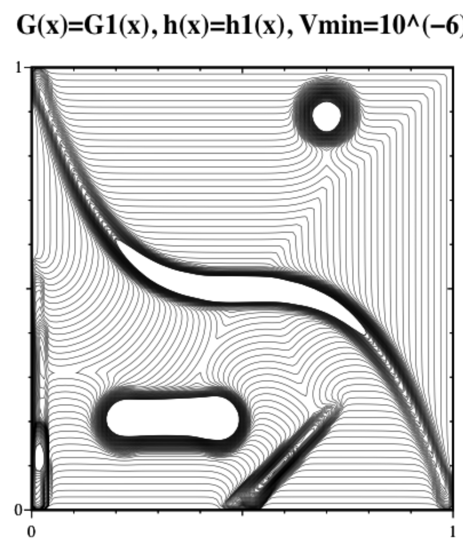


Figure 4d

5.2 Here we present some meshes

The reference length *dgr* and the other parameters are defined below or in Section 5.1.

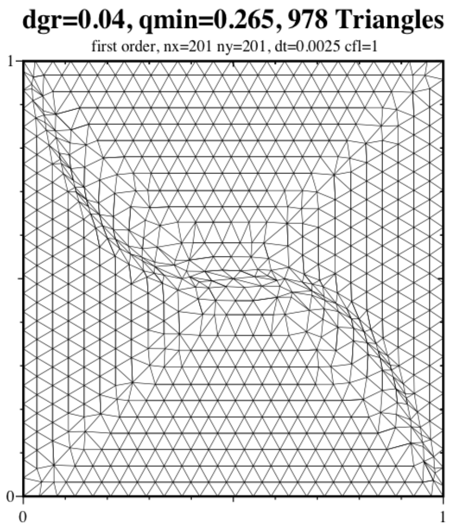


Figure 5a

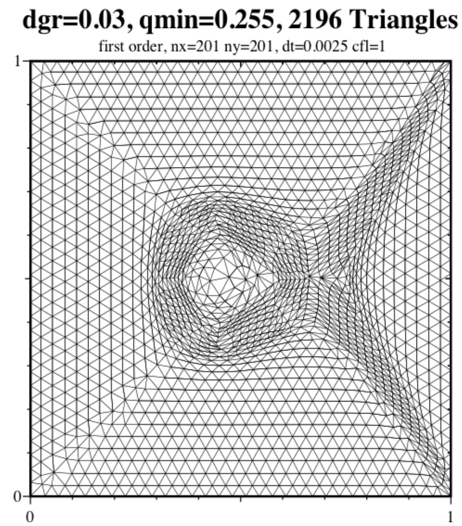


Figure 5b

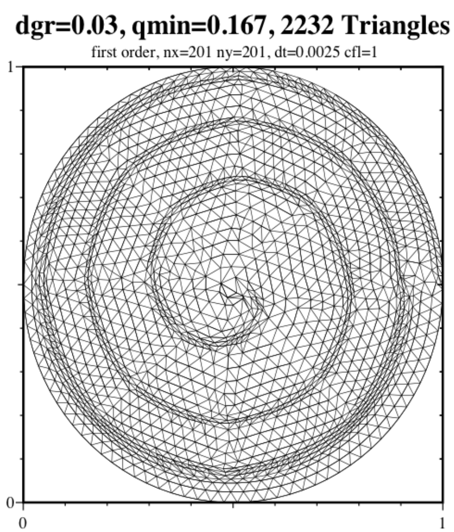


Figure 5c

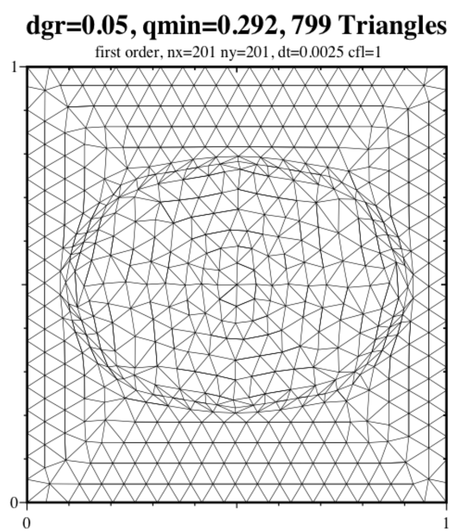


Figure 5d

Law of refraction : Small experiment - History/Motivation

Minimizing travel time ($\tau = \int$ dist/speed) in a two layered medium.

$$\frac{\sin\phi_1}{c_1} = \frac{\sin\phi_2}{c_2}$$

History (Historia Mathematica 10 (1983) 48-62 *The Mathematical Technique in Fermat's deduction of the Law of refraction*. K. Andersen.)

- 1637 Descartes *Dioptrique* Assuming light goes faster in denser media.

- 1657 (letter to C. De la Chambre) principle of least time. application of his method to find minima : *Methodus ad disquirendam et minimam* Fermat. "adequation"

$$f(a+e) \simeq f(a)$$

then simplify, divide by e and finally "equate". *he obtained expressions with "an irregular and fantastic proportion" after long and tedious calculations and "his natural inclination towards indolence" had made him stop.*

- 1662, completes computations. Finds Descartes's law !

- (Differential Calculus) Leibnitz (1684) and Newtown (1687)

The optical length of a Path

Fermat principle says Rays are the path Y that minimize (actually extremize) traveltime

$$\phi(Y(T)) = \int_0^T n(Y(s))|\dot{Y}(s)|ds + \phi(Y(0))$$

$(n = \frac{1}{c})$

Simplify to 1+1 D problem (time + space) : $Y(s) = (t, y(t))$ (paraxial assumption)

Fix t, x,

$$\min_{\{y \in \mathcal{C}^\infty,\ y_0 \in \mathcal{R}\ y(t)=x\}} F(y, y_0) = \int_0^t L(s, y(s), \dot{y}(s))ds + \phi^0(y_0)$$

with $L(t, y, v) = n(t, y(t))\sqrt{1 + v^2(t)}$.

Classical problem (*L.C. Young, Lecture on the Calculus of Variations*) important hypothesis : strict convexity of $v \to L(t, y, v)$.
Existence : (hints)

First variation vanishes

Euler-Lagrange equations :
the Frechet derivative satisfy $F'(y, y_0) = 0$

$F(y+h, y_0+h_0) - F(y, y_0) = \int_0^t L(s, y+h(s), \dot{y+h}(s)) - L(s, y(s), \dot{y}(s))ds + \phi^0(y_0 +$

$= \int_0^t L_x(s, y(s), \dot{y}(s))h(s) + L_v(s, y(s), \dot{y}(s))\dot{h}(s)ds + O(|h|^2) + \phi_x^0(y_0)h_0 + O(h_0^2)$

$|h| = \sup_{s\in[0,t]}\{|h(s)| + |\dot{h}(s)|\}$

$= \int_0^t\{L_x(s, y(s), \dot{y}(s)) - \frac{d}{ds}L_v(s, y(s), \dot{y}(s))\}h(s)ds + O(|h|^2) + ...$

$\{L_v(0, y(0), \dot{y}(0)) - \phi_x^0(y_0)\}h_0 + O(h_0^2).$

$$< \frac{\partial F}{\partial y}(y_0, y), h >= \int_0^t \{L_x(s, y(s), \dot{y}(s)) - \frac{d}{ds} L_v(s, y(s), \dot{y}(s))\} h(s) ds$$

$$\frac{\partial F}{\partial y_0}(y_0, y)\, h_0 = L_v(0, y(0), \dot{y}(0)) - \phi_x^0(y_0)$$

$$\begin{cases} L_x(s, y(s), \dot{y}(s)) - \frac{d}{ds} L_v(s, y(s), \dot{y}(s)) = 0 \\ L_v(0, y(0), \dot{y}(0)) - \phi_x^0(y_0) = 0. \end{cases}$$

Recap. on the Legendre Fenchel transform ($L_{vv} > 0$)

Define

$$H(t,x,p) = L^{\star}_{v\to p}(t,x,p) = \sup_{v\in R^d}\{p.v - L(t,x,v)\}$$

Then $L(t,x,v) = L^{\star\star} = \sup_{p\in R^d}\{p.v - H(t,x,p)\}$

and $H_p = (L_v)^{-1}$ i.e.

$$v = H_p(t,x,p) \leftrightarrow p = L_v(t,x,v)$$

and are the optimal args, and for (v,p) linked as above we have

$L(t,x,v) = L_v(t,x,v).v - H(t,x,L_v(t,x,v)) = p.H_p(t,x,p) - H(t,x,p))$

and

$$L_x(t,x,v) = H_x(t,x,p)$$

Examples

$$L(v) = \frac{1}{2}v^2 \rightarrow H(p) = \frac{1}{2}p^2$$

(Burgers equation)

$$L(t,x,v) = n(t,x)\sqrt{1+v^2} \rightarrow H(t,x,p) = -\sqrt{n^2(t,x) - p^2}$$

(Eikonal equation)

The Hamiltonian system

Set

$$p(s) = L_v(s, y(s), \dot{y}(s)),$$

which automatically gives

$$\dot{y}(s) = H_p(s, y(s), p(s)), \quad y(0) = y_0 \ (by\ construction).$$

take the time derivative of p

$$\dot{p}(s) = -H_x(s, y(s), p(s)), \quad p(0) = \phi_x^0(y_0), \quad (Euler - Lagrange).$$

also remark that $\phi(y(t)) = \int_0^t L(s, y(s), \dot{y}(s))ds + \phi(y_0)$ satisfies

$$\dot{\phi}(s) = H_p(t, y(s), p(s)).p - H(s, y(s), p(s)), \quad \phi(0) = \phi^0(y_0)$$

Back to RT

In practice y, p, ϕ as functions of (t, y_0)

Apply $H(t,x,p) = -\sqrt{n^2(t,x) - p^2}$

$$\begin{cases} \dot{y}(s) = \frac{p}{\sqrt{n^2 - p^2}} & y(0, y_0) = y_0 \\ \dot{p}(s) = -\frac{n_x n}{\sqrt{n^2 - p^2}} & p(0, y_0) = \phi_x^0(y_0) \\ \dot{\phi}(s) = \frac{n^2}{\sqrt{n^2 - p^2}} & \phi(0, z_0) = \phi^0(y_0) \end{cases}$$

Setting $q = \sqrt{n^2 - p^2}$ and $\dot{z}(s) = q$ and then change parameterization $dz = ds\sqrt{n^2 - p^2}$ to recover our original ray tracing equations for $\{Y = (y, z),\ P = (p, q)\}$.

Lagrangian $\rightarrow$ Eulerian : Classical solutions

Consider $\Omega_T = \{y(t, y_0), \; \forall (t, y_0) \in [0, T] \times R^d\}$.

As long as $J(t, y_0) = det(\frac{\partial y(t,y_0)}{\partial y_0}) \neq 0$. One can invert

$$y_0 \rightarrow x = y(t, y_0)$$

Then define $\psi(t, x)$ as

$$\psi(t, y(t, y_0)) = \phi(t, y_0).$$

ψ is smooth and

$$\psi_x(t, y(t, y_0)) = p(t, y_0).$$

The Hamilton-Jacobi equation

Time differentiate :

$$\partial_t \psi(t, y(t, y_0)) + H_p(t, y(t, y_0), p(t, y_0)) \cdot \nabla_x \psi(t, y(t, y_0)) =$$

$$p(s, y_0) \cdot H_p(s, y(s, y_0), p(s, y_0)) - H(s, y(s, y_0), p(s, y_0)),$$

Write in Eulerian coordinates

$$\begin{cases} y(t, y_0) \to x \\ p(t, y_0) = \nabla_x \psi(t, y(t, y_0)) \to \nabla_x \psi(t, x) \end{cases}$$

We get

$$\partial_t\psi(t,x) + H(t,x,\nabla_x\psi(t,x)) = 0, \;\; \psi(0,x) = \phi^0(x).$$

Rem : $\partial_t\psi > 0$ and $H = -\sqrt{n^2 - p^2}$ (paraxial hypothesis), we find the original 2-D the Eikonal equation.

Classical solutions break down at caustics

Looking around a Caustic point : $\{y^c = y(y_o^c), p^c = p(y_o^c)\}$.

Fold :

$$\begin{cases} \dfrac{\partial y}{\partial y_o} = 0 \\ \dfrac{\partial^2 y}{\partial^2 y_o} \neq 0 \end{cases}$$

Then (TE) :

$$y(y_o) = y^c + 0 + \frac{(y_o - y_o^c)^2}{2} \frac{\partial^2 y}{\partial^2 y_o}(y_o^c) + O\{(y_o - y_o^c)^3\}$$

Cusp :

$$\begin{cases} \frac{\partial y}{\partial y_o} = 0 \\ \frac{\partial^2 y}{\partial^2 y_o} = 0 \end{cases}$$

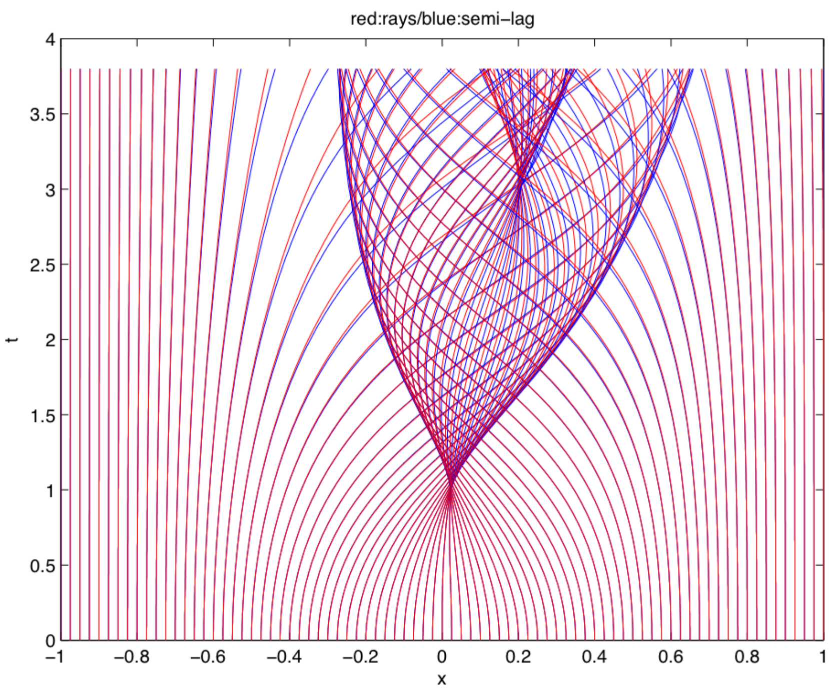
Then :

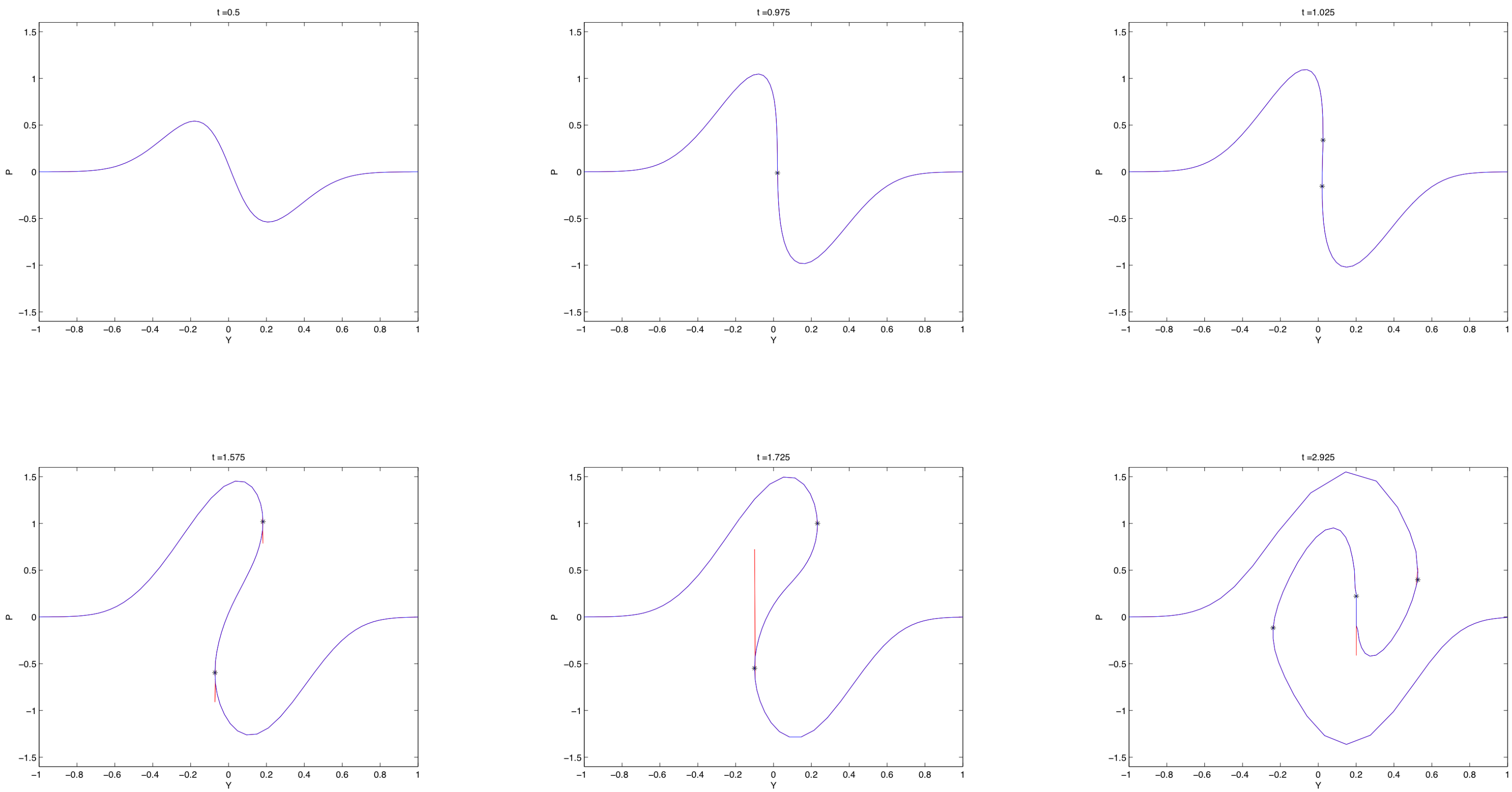
$$y(y_o) = y^c + 0 + 0 + \frac{(y_o - y_o^c)^3}{6} \ \frac{\partial^3 y}{\partial^3 y_o}(y_o^c) + O\{(y_o - y_o^c)^4\}$$

An example :

$n(t,x) = 2.8$ if $|x - 0.2 \sin 0.3\,t| \geq 1$ and $n(t,x) = 2.8 + 0.4 * \exp^{-10\,(x-0.2\,\sin 0.3\,t)^2}$ else,

$\phi_o \equiv 0$ as initial condition.




t =0.5
t =0.975
t =1.025
t =1.575
t =1.725
t =2.925
P
Y

Basics on viscosity solutions

G. Barles, Solutions de viscosités des Equu. de H,-J., Springer 1994
P.L. Lions, Generalized solutions of H.-J. Equations, Pitmann
Some good ressources online , in French though ... .

`www.math.psu.edu/bressan/PSPDF/simy.pdf`

Has evolved into a very general theory of existence and approximation for non-linear equations.

$$F(x, u(x), Du(x), D^2u(x)) = 0$$

$F(x, u, P, M)$ Needs :

- Uniform continuity. - (Strict) convexity in p.
- Coercivity in p. - Ellipticity in M.

Consider simplified Pbm

$$\epsilon u + H(x, Du) = 0 \; x \in \mathcal{R}^d$$

(same with Dirichlet/Neumann or Cauchy problems).

<u>Defs :</u>

- $u(x) \in \mathcal{C}(\mathcal{R}^d)$ is a viscosity sub(super)-solution if $\forall \phi \in C^1$

  x_0 is a point of max. (min.) of $u - \phi$

  $\Rightarrow \epsilon u + H(x, D\phi) \leq \ (\geq)\ 0$

- u is a viscosity solution iff sub- and super-solution.

First Remarks

- One can restrict to ϕ s.t. $u(x_0) = \phi(x_0)$.

- One can restrict to strict min/max assumption : $\phi = \phi + \alpha|x - x_0|^2$.

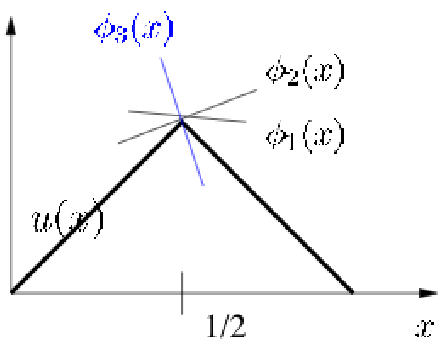
- Classical solutions are viscosity solutions.

Exemple (1-d)

$$|u'| = 1, \;\; u(0) = u(1) = 0 \;\; H(u') = |u'| - 1$$

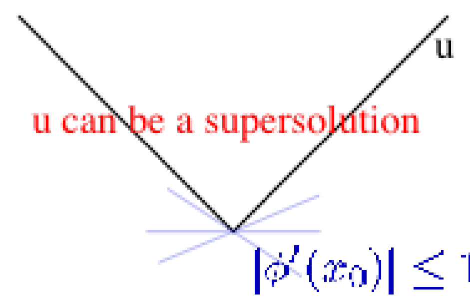
Has no classical solutions.
Has many "generalized" $\mathcal{W}^{1,\infty}$ solutions.
Has a unique $\mathcal{BUC}$ viscosity solution.

Viscosity solution allows upward (but not downward) kinks :



ϕ_1 and ϕ_2 are two possible functions;
ϕ_3 can't verify $u(x) \leq \phi$ for every neighbourhood of $\frac{1}{2}$.

Figure 2.2: Illustration of: $\forall x \in V, u(x) \leq \phi(x) \Longrightarrow |\phi'(x)| \leq 1$



Properties

maximum principle...

Prop. (Unicity) : Let $\overline{u}$ (resp. $\underline{u}$) be a l.s.c. (resp. u.s.c) super- (resp. sub-) viscosity solution of then $\underline{u} \leq \overline{u}$.

Evanescent viscosity : $\forall \delta > 0$,

$$\begin{cases} -\delta \Delta u + \epsilon u + H(x, Du) = 0 \; x \in \Omega \\ \end{cases}$$

has a unique $\mathcal{C}^2 \cap \mathcal{W}^{1,\infty}$ solution u_δ which converges to the viscosity solution ($\epsilon = 0$).

(Sketch of the proof).

Back to the Calculus of Variation and Cauchy Pbm Consider

$$\begin{cases} \partial_t u + H(x, Du) = 0 \ t > 0, \ x\mathcal{R}^d \\ u(0,x) = u_0(x) \end{cases}$$

Set $L = H^\star$ (see recap. on Legendre Transform) and

$$v(t,x) = \inf_{y_0}(u_0(y_0) + \inf_{\substack{\gamma \in \mathcal{W}^{1,\infty}(0,t) \\ \gamma(t) = x \\ \gamma(0) = y_0}} \int_0^t L(t, \gamma, \dot{\gamma}) dt)$$

Then v is the unique $\mathcal{BUC}$ and Lipschitz viscosity solution of the HJ equ.

Sketch

- Inf is reached by $\gamma \in \mathcal{C}^2$, $|\dot{\gamma}| \leq C_t$.
- v super-solution (inf prop.)
- v sub solution (pontryagin).
- Lipschitz.

$+ \ Du = L_v$

Intro. to explicit formula (Lax-Oleinik

Let $H(p)$ be convex in and depending only on p. Remember $L = H^\star$, then the viscosity solution of

$$\partial_t u + H(u_x) = 0\,,\ u(0,x) = u_0(x)$$

can be written as (backward parameterization)

$$u(t,x) = \inf_v \{u_0(\gamma(t)) + \int_0^t H^\star(v)ds\}$$

where $\dot\gamma = -v,\ \gamma(0) = x$. Then use Jensen inequality

$$\int_0^t H^\star(v)ds \geq tH^\star(\frac{1}{t}\int_0^t vds) = tH^\star(\frac{x-\gamma(t)}{t})$$

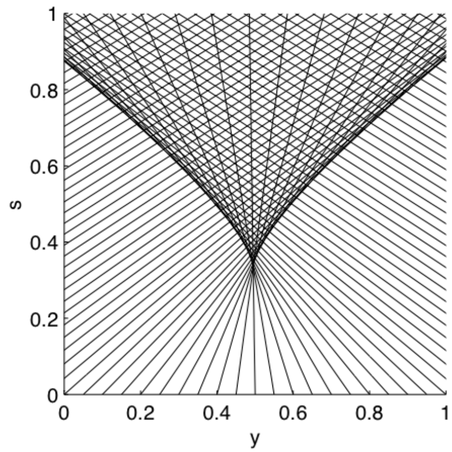
i.e. straight lines from $x \to y = \gamma(t)$ are optimal. We can rewrite the viscosity solution as :

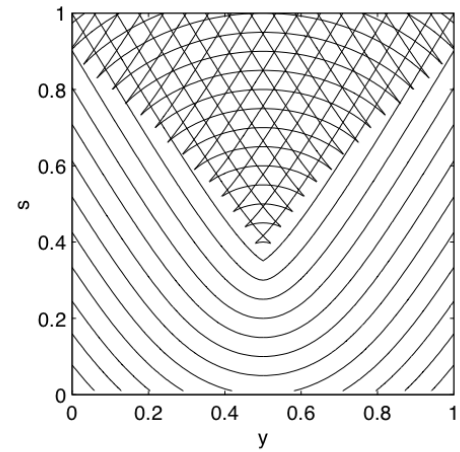
$$u(t,x) = \inf_y \{u_0(y) + tH^\star(\frac{x-y}{t})ds\}$$

(Lax Oleinik formula).

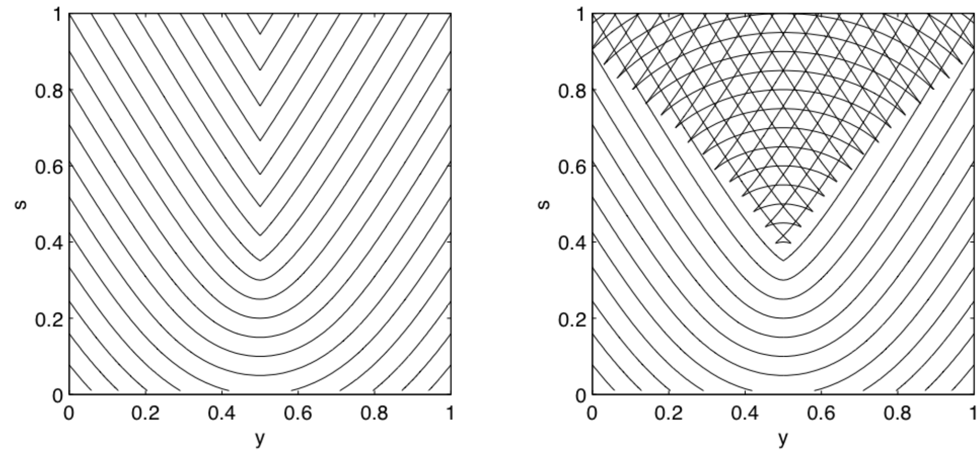
For ex. $H(p) = -\sqrt{1-p^2} \to H^\star(v) = \sqrt{1+v^2}$ gives the distance funtion from the Cauchy data.

An example $n \equiv 1$:





An example $n \equiv 1$:



Link with conservation laws in 1D. Setting formally $u = \phi_x$ yields

$$\partial_t \phi + H(\phi_x) = 0 \Leftrightarrow \partial_t u + (H(u))_x = 0$$

Ex. : Burgers $H(p) = \frac{|p|^2}{2}$

Rem. 1 : H.-J. is similar to Homegenous GO.

Rem. 2 : Inf principle of the Viscosity solution is equivalent to R.H. condition. Shock speed : $s = \frac{u^+ + u^-}{2}$

Back to RT (numerics)

$$\begin{cases} \frac{dY}{ds} = P(s) \\ \frac{dP}{ds} = \frac{1}{2}\nabla n^2(Y(s)) \end{cases}$$

The phase Φ, can be computed as the integral of $\|P\|^2$ along a ray $Y(s)$, since

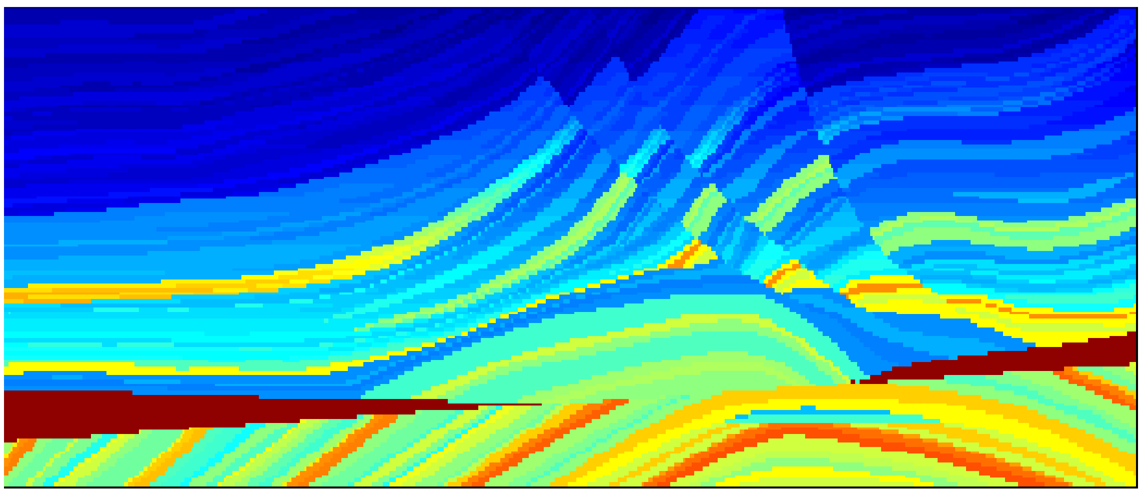
$$\frac{d}{ds}\Phi(Y(s)) = \frac{dY}{ds}.\nabla\Phi(Y(s)) = \|P(s)\|^2 = n^2(Y(s))$$

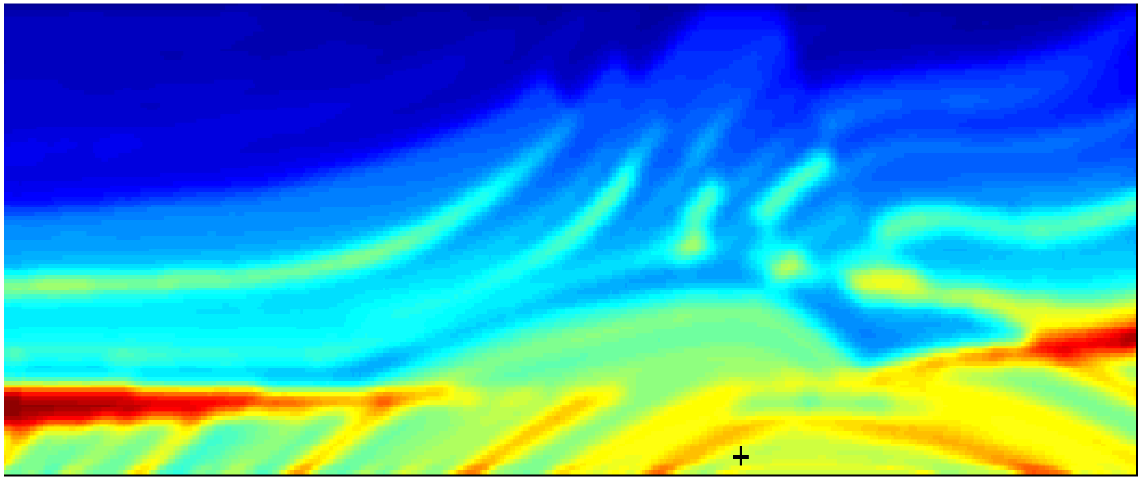
In practice solve for a family of rays parameterized by Y_0 or θ. Generally with a RK or adapative RK method.

note : should say a word on symplectic solvers...

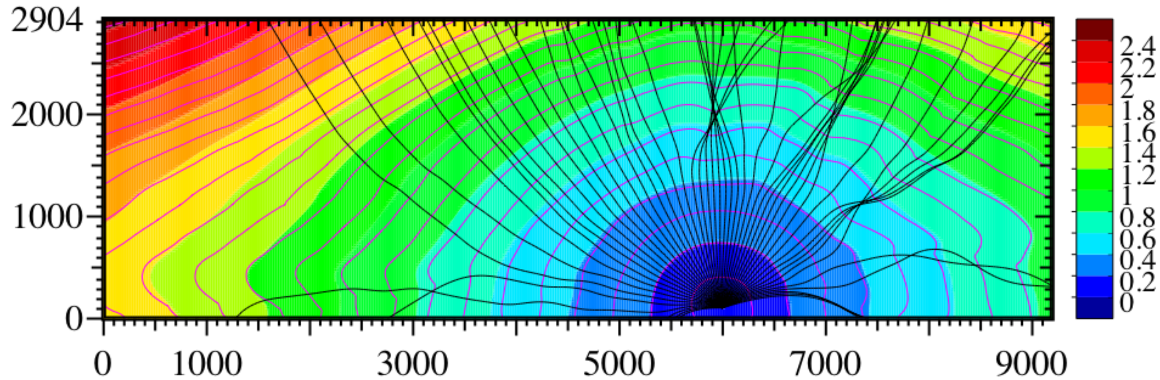
Does it work ? : RT in Marmousi

http://www-rocq.inria.fr/ benamou/testproblem.html





Marmousi Model



Dynamical (or Paraxial Ray tracing)

Ray methods in Seismology, Cerveny, Molotkov, Pscencik, Charles U. Praha (77) ....
G. Lambaré 2002 GO++ winter school notes

Idea : Linearize RTS (and change notations :$(\delta x, \delta p)$: first order variation in $(x, p) = (Y, P)$)

$$\begin{cases} \frac{d\delta x}{ds} = \delta p \\ \frac{d\delta p}{ds} = \nabla(\frac{1}{2}\nabla n^2(x))\delta x \end{cases}$$

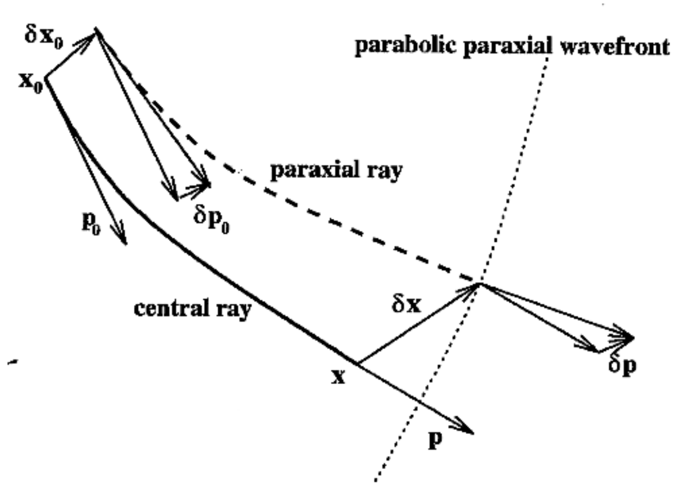


Figure 2.4: *Paraxial approximation arround a central ray [Lambaré et al., 1996].*

Propagator matrix

Because this is linear in the initial conditions, one can factor various computations in a "propagator matrix"

$$(\delta x, \delta p)(s) = (\delta x, \delta p)(s_0)\mathcal{P}(s, s_0)$$

(transpose)

$$\frac{d\mathcal{P}}{ds} = \begin{pmatrix} 0 & Id \\ \nabla(\frac{1}{2}\nabla n^2(x)) & 0 \end{pmatrix} \mathcal{P}, \;\; \mathcal{P}(s_0, s_0) = Id$$

Rem. 1 : this can be computed very accurately along each ray.

Rem 2. : this provide a 2nd order accurate estimate of the Lagrangian submanifold $\{(x(s, y_0), p(s, y_0)) \,|\, y_0 \in ...\}$.

$x(y_0 + \delta y_0, p_0 + \delta p_0) = x(y_0, p_0) + ...$

Also give second order derivatives of travel time ... discuss ray coordinates ...

Wavefront construction

Main idea : maintain ray density through interpolation (adaptative gridding). Need a good criterium.

Problem : stretching or concentration may also may also occur in the p dimensions.

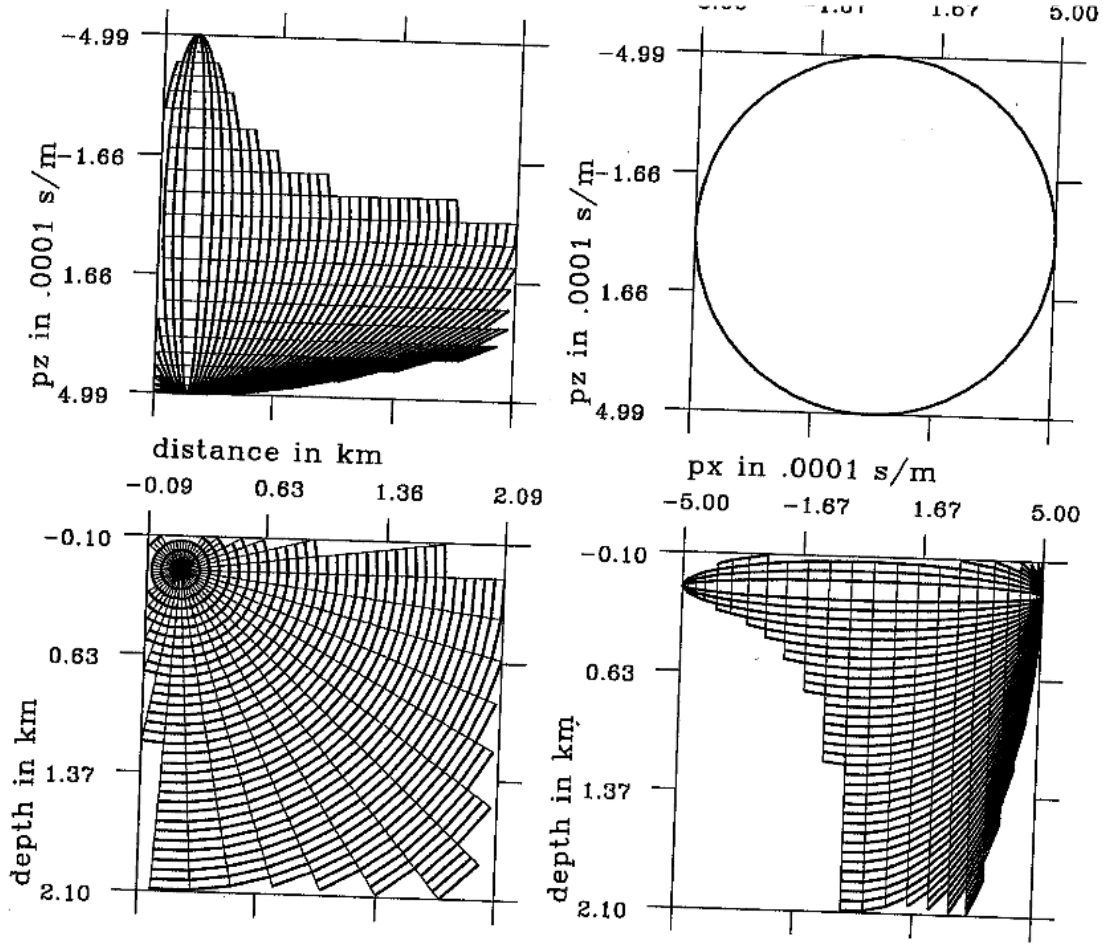


Figure 3.3: *The Lagrangian manifold for a constant velocity field (v=2km/s). The paving of the Lagrangian manifold is shown for various 2-D projections of the phase space* $[(x,z),(x,p_x),(p_x,z),(p_x,p_z)]$. *The point source is a x=110 m and z=110 m. The travel time step is .03 s, and the ray density criterion is* $dx_{max} = 10$ *m and* $dp_{max} = 10 \times 10^{-6} sm^{-1}$. *In the plane* (x,z) *the rays are straight lines and the wave-front are circles [Lambaré et al., 1996].*

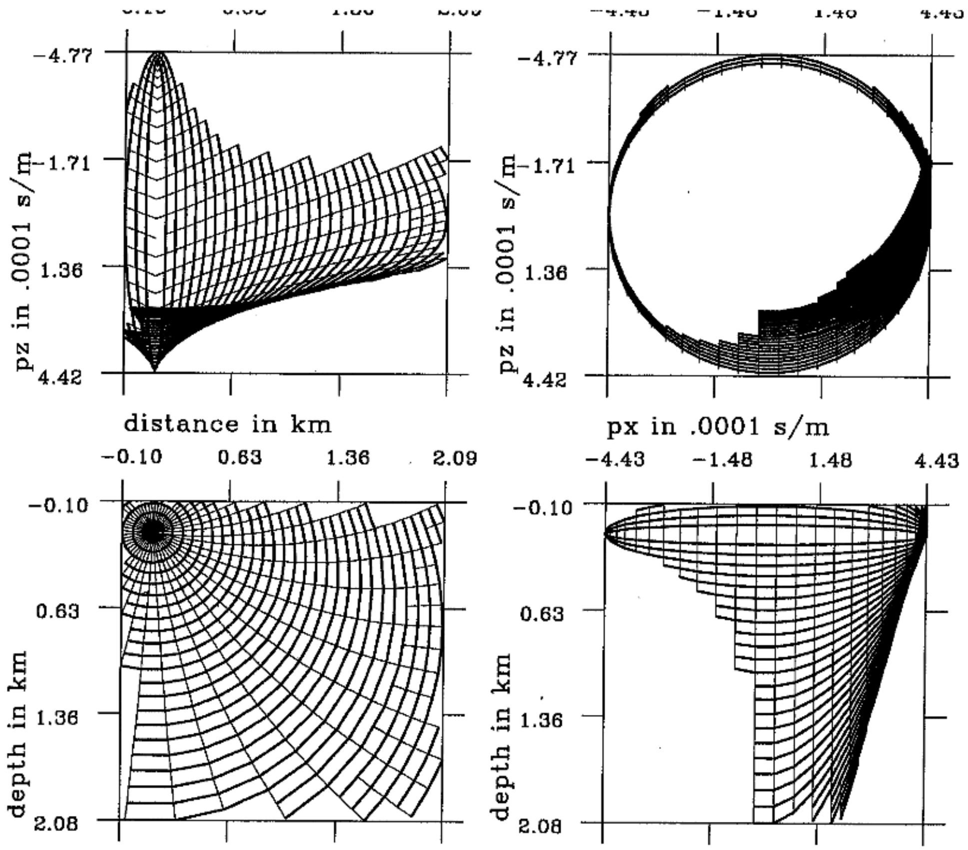


Figure 3.4: *The Lagrangian manifold for a constant gradient velocity field. Velocity increases linearly with depth, from 2083 m/s at z=0 m to 3833 m/s at z=2000 M. The paving of the Lagrangian manifold is shown for various 2-D projections of the phase space* $[(x,z),(x,p_x),(p_x,z),(p_x,p_z)]$. *The point source is a x=110 m and z=110 m. The travel time step is .03 s, and the ray density criterion is* dx_{max} = 10 *m and* $dp_{max} = 10 \times 10^{-6} sm^{-1}$. *In the plane* (x,z) *the rays and the wavefront are circles. Note a p-caustic atthe bottom of the* (x,p_z) *projection [Lambaré et al., 1996].*

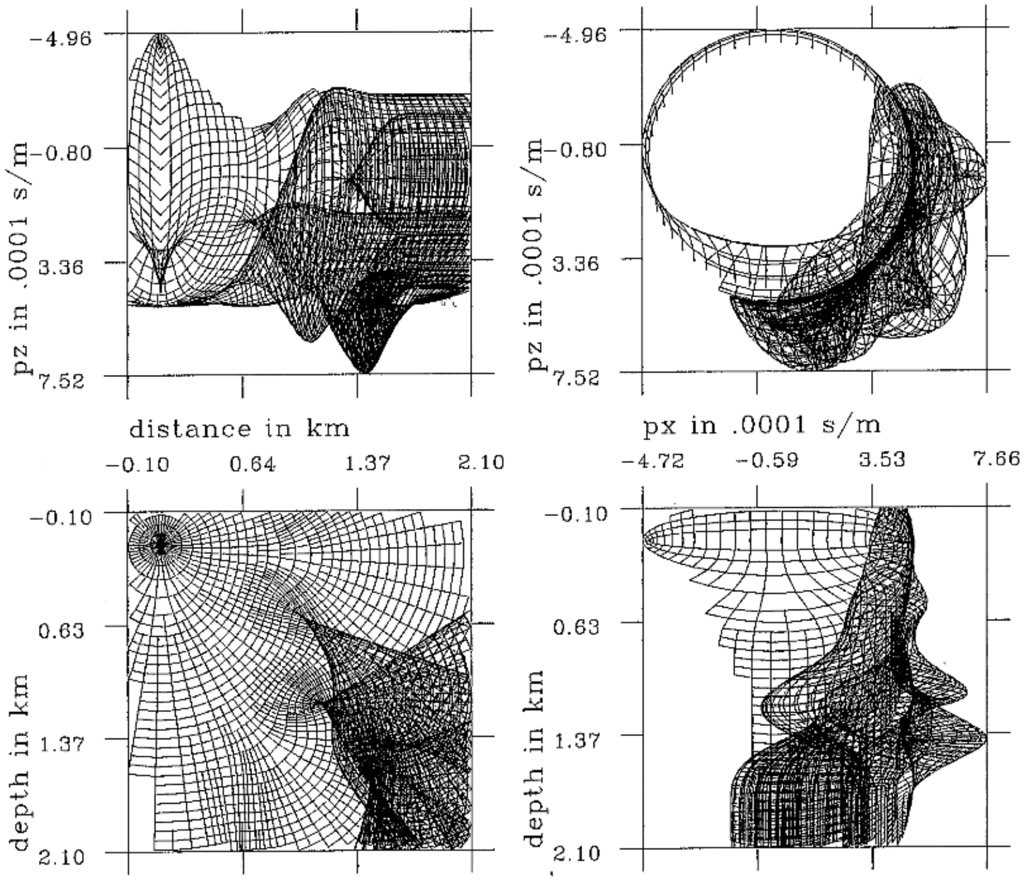


Figure 3.6: *The Lagrangian manifold for a complex velocity field. The paving of the Lagrangian manifold is shown for various 2-D projections of the phase space* $[(x,z),(x,p_x),(p_x,z),(p_x,p_z)]$. *The point source is a x=110 m and z=110 m. The travel time step is .03 s, and the ray density criterion is* $dx_{max} = 10$ *m and* $dp_{max} = 10 \times' 10^{-6} sm^{-1}$. *The ray field exhibits many caustics, and projections involve many overlapping cells. the density of rays in the configuration space is increased in th eregions of strong curvature of the rays [Lambaré et al., 1996].*

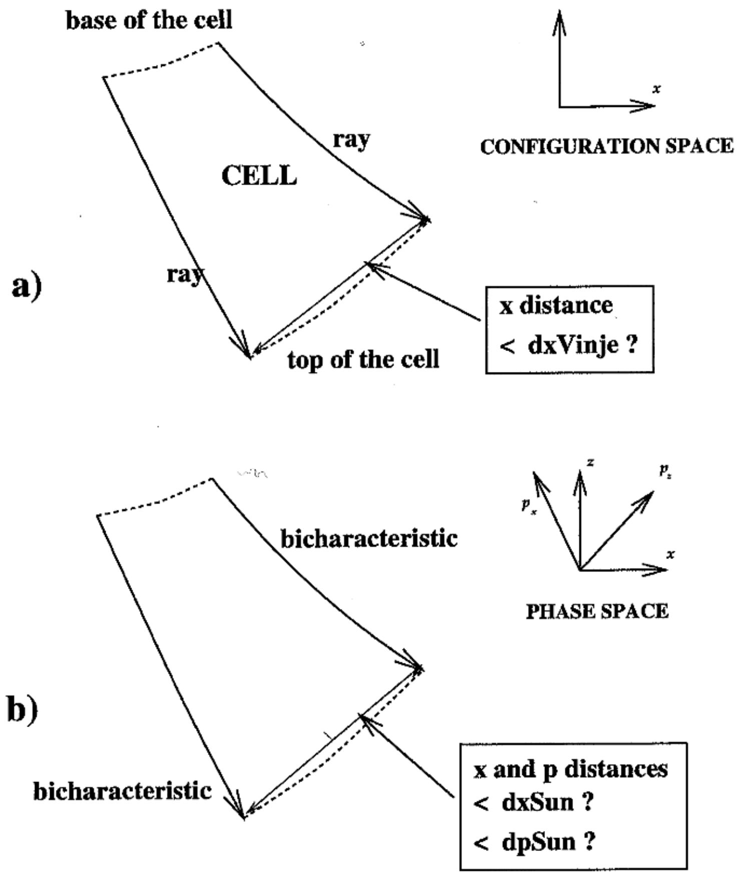


Figure 3.7: *Vinje's and Sun's criteria for checking the size of the cells. (a) Vinje's criterion, where the* **x** *distance of the top of the cell must not exceed the value* dx_{Vinje}. *(b) Sun's criterion, where the* **x** *and* **p** *distances of the top of the cell must not exceed the values* dx_{Sun} *and* dp_{Sun}. *[Lambaré et al., 1996].*

Sol 1 : *Vinje ...* x distance. Problem stretching or concentration may also may also occur in the p dimensions.

Sol 2 : *Sun ...* use x and p distance.

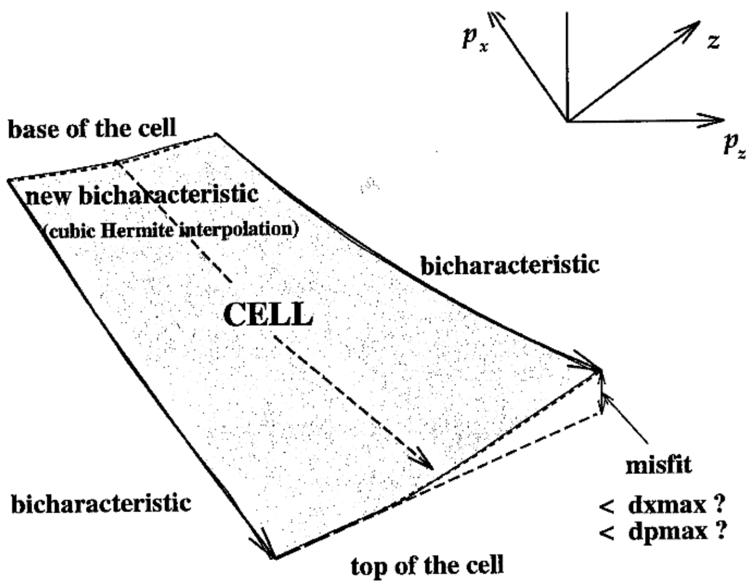


Figure 3.8: *Uniform ray density criterion for checking the size of the cells. The misfit between the tangent plane (defined by the paraxial approximation) and the exact manifold must not exceed a given value in distance and slowness. A new bicharacteristic is interpolated at the base of the cell if the misfit exceeds* (dx_{max}, dp_{max}) *[Lambaré et al., 1996].*

Sol 3 : *Lambaré et al ...* Use paraxial quantities to estimate error.

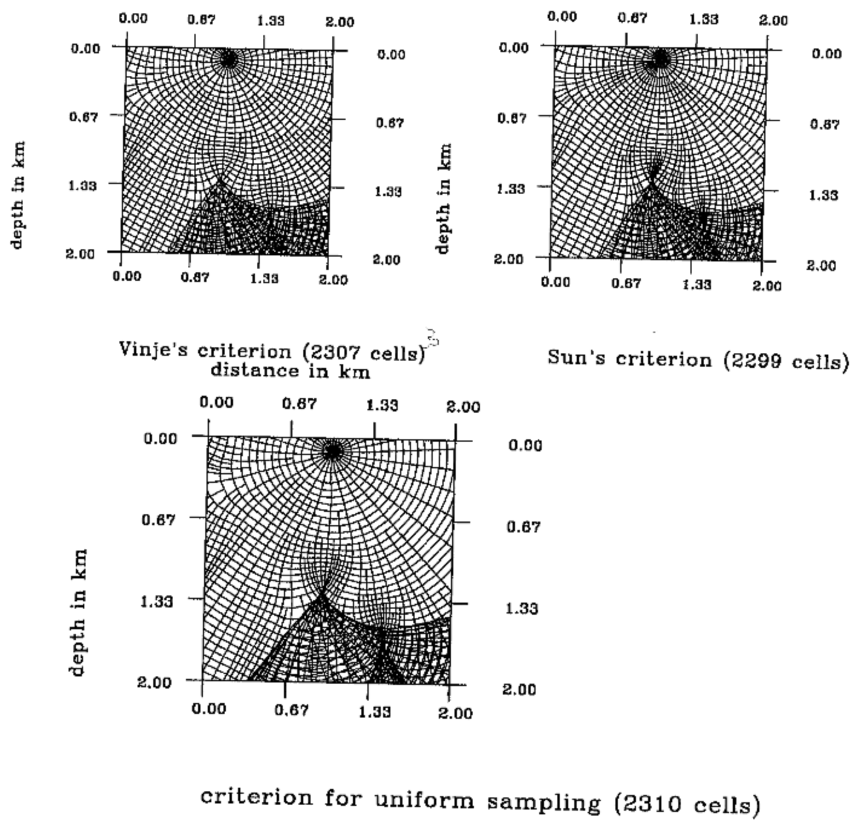


Figure 3.11: *Comparaison of ray density criteria on the complex velocity model presented on Figure 3.5. The source is a at x=1000m, z=1100m. The travel time step is .03 s. The ray field is sampled according to : (a) Vinje's criterion; (b) Sun's criterion; (c) Uniform ray density criterion. In order to compare equivalent results in terms of computational cost, we choose the values of the ray-density criteria in such a way as to have the same number of cells (about 2305 in each case) [Lambaré et al., 1996].*

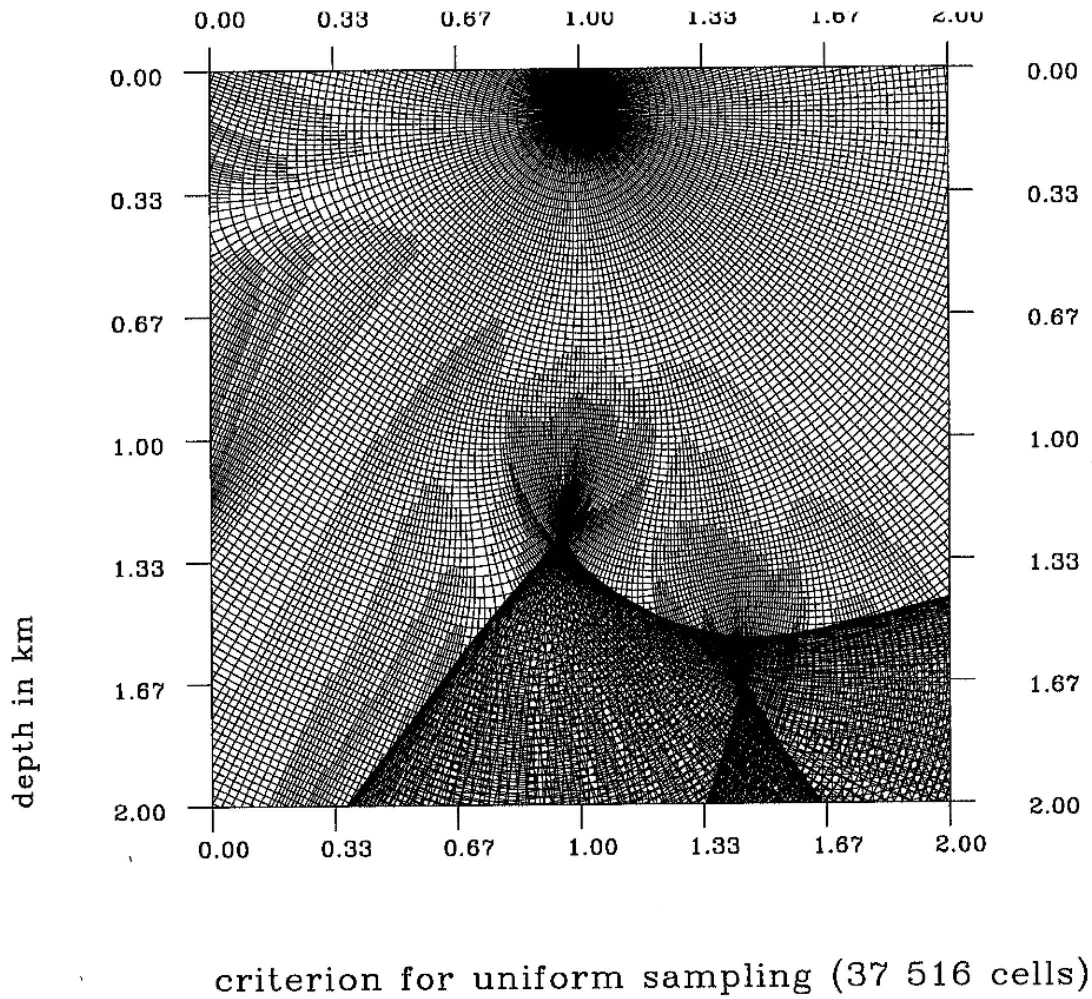


Figure 3.12: *Reference ray-field sampling for testing the accuracy of the ray field interpolation. The traveltime step is 0.01 s,* $dx_{max} = 1$ *m,* $dp_{max} = 10^{-6}$ *s/m . 35 516 cells were generated [Lambaré et al., 1996].*

Intro to Eulerian FD upwinding

Consider the simplest non trivial HJ equ. $(c > 0)$

$$u_t + c u_x = 0$$

i.e. $H(p) = cp$ (note $L(v) = 0$ for $v = c$ and ∞ else) with explicit solution $u(t,x) = u_0(x - ct)$. Then discretize : $u_j^k = u(k\,dt, j\,dx)$ and "upwind". Two solutions

$$\begin{cases} u_j^{k+1} = u_j^k + dt \frac{u_{j+1}^k - u_j^k}{dx} \\ u_j^{k+1} = u_j^k + dt \frac{u_j^k - u_{j-1}^k}{dx} \end{cases}$$

Remarks

- Right upwinding does not converge (try $u_0 = 0$ for $x > 0$ and $\neq 0$ for $x < 0$.

- Numerical propagation speed is $\frac{dt}{dx}$. consider sequences s.t. $(k\,dt, j\,dx) \to (t,x)$ as $(dt,dx) \to 0$ and define
$$\tilde{c} = \limsup \frac{dx}{dt}$$
Then no convergence (wrong dependance) if $\tilde{c} < c$. This is the sense of the CFL condition :
$$c\frac{dt}{dx} \leq 1$$

- Left Upwinding can be rewritten

$$u_j^{k+1} = (1 - c\frac{dt}{dx})u_j^k + c\frac{dt}{dx}u_{j-1}^k$$

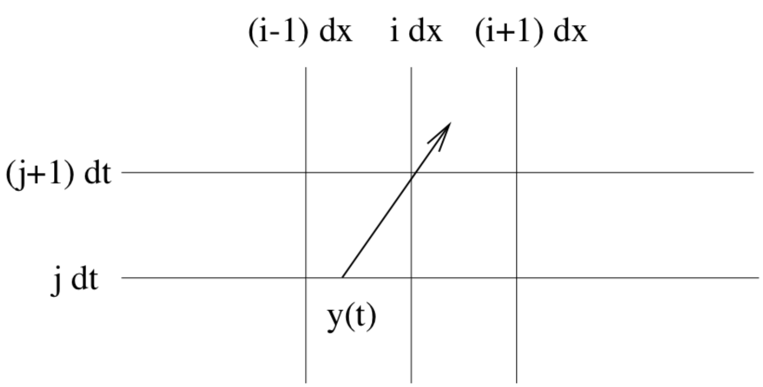
Scheme is monotone (max principle) under CFL.

Back To H.J in 1+1 D

Take (for instance) $H(p) = -\sqrt{1-p^2}$ and look a a time step $[t_j, t_{j+1}]$ around x_i.

Assume further that the phase ψ is piewise linear with left and right slopes $\psi_x^{\mp}$.

Let $y(t)$ be a ray beween (t_j, x) and (t_{j+1}, x_i) such that $x = y(t_j) < x_i$.



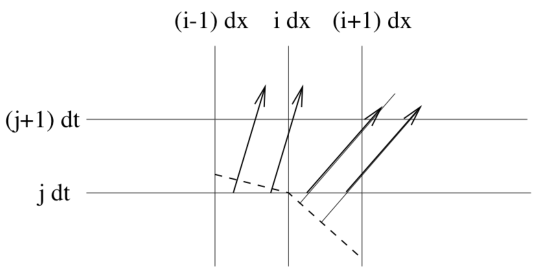
Integrate the Lagrangian phase (Riemman problem)

$$
\begin{aligned}
\psi(t_{j+1}, x_i) = \phi(t_{j+1}) \quad &= \phi(t_j) + \int_{t_j}^{t_{j+1}} p(s) H_p(s, y(s), p(s)) - H(p(s))ds \\
&= \phi(t_j) + \psi_x^-(x_i - y(t_j)) - dtH(\psi_x^-) \\
&= \psi(t_j, x_i) - dtH(\psi_x^-)
\end{aligned}
$$

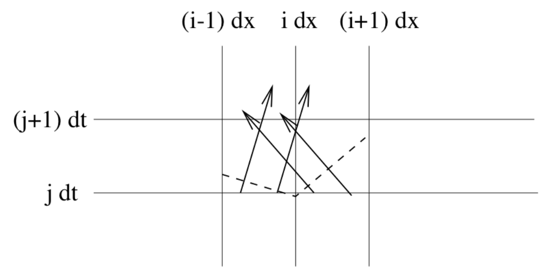
Looks like a FD scheme for the H.-J. equ.

$$\psi_i^{j+1} = \psi_i^j - dtH(\psi_x^-)$$

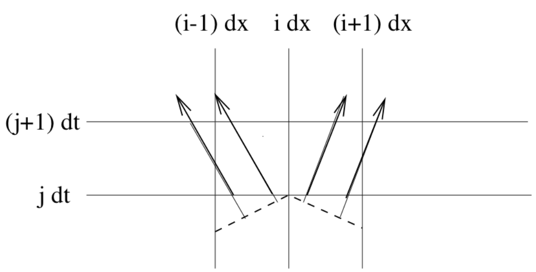
Upwinding : Minimize traveltime ...



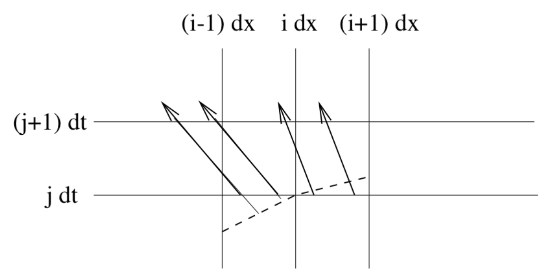
A



B



C



D

A. $\psi_x^- \geq 0,\ \psi_x^+ \geq 0.$
B. $\psi_x^- \geq 0,\ \psi_x^+ < 0.$
C. $\psi_x^- < 0,\ \psi_x^+ \geq 0.$
D. $\psi_x^- < 0,\ \psi_x^+ < 0.$

the "Godunov" scheme

$$\partial_t \psi_i(t) = -H(t, x_i, \psi^u_{x,i}(t)) = 0, \quad \psi_i(0) = \phi^0(x_i).$$

$$\begin{cases} \psi^u_{x,i}(t) = \max((\psi^l_{x,i}(t))^+, (\psi^l_{x,i}(t))^-) \\ \psi^l_{x,i}(t) = \dfrac{\psi_i(t) - \psi_{i-1}(t)}{dx}, \quad \psi^r_{x,i}(t) = \dfrac{\psi_{i+1}(t) - \psi_i(t)}{dx}. \end{cases}$$

$(p)^+ = max(p, 0)$ and $p^- = min(p, 0)$.

Note on B.C. I.C CFL bound on H_p ....

Generalization 1. (convex H with min at $p = 0$)

Recall $H(t, x, p) = \sup_v \{p.v - L(t, x, v)\}$ and $L = H^\star$ also convex with min at $v = 0$ ($H_p = (l_v)^{-1}$) .

Remark that (omit (t, x)) $H(p) = \max(H^+(p), H^-(p))$ with

$$\begin{cases} H^+(p) = \sup_{v \geq 0}\{p.v - L(v)\} = H(p^+) \\ H^-(p) = \sup_{v \leq 0}\{p.v - L(v)\} = H(p^-) \end{cases}$$

H.-J. becomes (upwinding)

$$\partial_t \psi_i(t) = \max(H^+(\psi^l_{x,i}(t)), H^-(\psi^r_{x,i}(t)))$$

Generalisation 2 (Godunov numerical Hamiltonian)

$$H^G(\psi_x^-, \psi_x^+) = Ext_{p \in I(\psi_x^-, \psi_x^+)} H(p)$$

where

$$Ext_{p \in I(a,b)} = \begin{cases} \min_{a<p<b} & \text{if } a \leq b \\ \max_{b<p<a} & \text{if } a > b \end{cases}$$

Lax-Friedrich

$$H^{LF}(t, x_i, \psi^+_{x,i}(t), \psi^-_{x,i}(t)) = H(t, x_i, \frac{\psi^+_{x,i}(t) + \psi^-_{x,i}(t)}{2}) - \alpha \frac{\psi^+_{x,i}(t) - \psi^-_{x,i}(t)}{dx}$$

α chosen to maintain monotonicity (derivative of H).

Higher Order (ENO, WENO ...), time integration ...

Recall ....

Recall $H(t,x,p) = \text{sup}_v\{p.v - L(t,x,v)\}$ and $L = H^\star$.

Remark that (omit (t,x)) $H(p) = \max(H^+(p), H^-(p))$ with

$$\begin{cases} H^+(p) = \text{sup}_{v \geq 0}\{p.v - L(v)\} \\ H^-(p) = \text{sup}_{v \leq 0}\{p.v - L(v)\} \end{cases}$$

H.-J. becomes (upwinding)

$$\partial_t \psi_i(t) = \max\{H^+(\psi^-_{x,i}(t)), H^-(\psi^+_{x,i}(t))\}$$

Time discretization $(.^k)$

$$\psi_i^{k+1} = \psi_i^k - dt * \max\{H^+(t^k, x_i, D^-\psi_i^k), H^-(t^k, x_i, D^+\psi_i^k)\}$$

where :

$$\begin{cases} D^l\psi_i^k = \frac{\psi_i^k - \psi_{i-1}^k}{dx} & D^r\psi_i^k = \frac{\psi_{i+1}^k - \psi_i^k}{dx} \\ H^+(p) = \sup_{v\geq 0}\{p.v - L(v)\} & H^-(p) = \sup_{v\leq 0}\{p.v - L(v)\} \end{cases}$$

Rewrite as :

$$\psi_i^{k+1} = \mathrm{min}\{U^- \psi_i^k, U^+ \psi_i^k\}$$

where :

$$\begin{cases} U^- \psi_i^k = \mathrm{inf}_{v \geq 0}\{dt\, H^\star(t^k, x_i, v) + \frac{dt}{dx}\, v\, \psi_{i-1}^k + (1 - \frac{dt}{dx}\, v)\, \psi_i^k\} \\ U^+ \psi_i^k = \mathrm{inf}_{v \leq 0}\{dt\, H^\star(t^k, x_i, v) - \frac{dt}{dx}\, v\, \psi_{i+1}^k + (1 + \frac{dt}{dx}\, v)\, \psi_i^k\} \end{cases}$$

Important remark : can restrain to $|v| \leq V = (\|H_p\|_\infty)$. The CFL condition can be written :

$$V \frac{dt}{dx} \leq 1$$

Fondamental properties of the numerical scheme

- Consistence : For a smooth function ϕ, looking at a sequence s.t. $(k\,dt, j\,dx) \to (t,x)$ as $(dt,dx) \to 0$ and setting $\psi_i^k = \phi(t^k, x_i)$, we get

$$\lim\{\psi_i^{k+1} - \mathrm{min}(U^-\psi_i^k, U^+\psi_i^k)\} = \partial_t\phi(t,x) - H(t,x,\phi_x(t,x))$$

- Monotone : Check that $(\psi_i^{k+1})_i$ is a monotone (increasing) function of $(\psi_i^k)_i$.

  Just need $1 \pm \frac{dt}{dx} v > 0$, CFL condition is enough for $U^{\pm}$.

- Stability : (under CFL) the discrete solution is uniformly bounded (independently of (dt, dx).)

$$\|U^{\pm}\psi_i^k|_\infty \leq dt\|H^\star\|_\infty + C\|\psi_i^k\|_\infty$$

so

$$\|\psi_i^{k+1}\|_\infty \leq dt\|H^\star\|_\infty + C\|\psi_i^k\|_\infty$$

is enough for finite time horizon.

Outline of convergence proof.

1. Using stability define :

$$\overline{\psi}(t,x) = \text{lim}_{(k\,dt, i\,dx) \to (t,x)} \sup \psi_i^k$$

$$\underline{\psi}(t,x) = \text{lim}_{(k\,dt, i\,dx) \to (t,x)} \inf \psi_i^k$$

   by construction $\underline{\psi} \leq \overline{\psi}$.

2. Show that (monotonicity + consistence) $\overline{\psi}$ is an upper semi continuous viscosity sub-solution.
   Show that (monotonicity + consistence) $\underline{\psi}$ is a lower semi continuous viscosity super-solution.

3. Then strong uniqueness garantees : $\overline{\psi} \leq \underline{\psi}$.

Back to full 2-D (x, y)

E. Rouy and A. Tourin. A viscosity solutions approach to shape-fr
SIAM J. Numer. Anal. **3** (1992) 867--884.

The Eikonal equation

$$\|\nabla\psi(X)\|^2 = n(X)^2 \quad X = (x, y)$$

can be written (optimal control formulation)

$$\sup_{\|Q\|\leq 1} \{\nabla\psi(x, y).Q - n(x, y)\} = 0.$$

Sup is reached for $q = \frac{\nabla\psi}{|\nabla\psi|}$ so can also use

$$\sup_{\|Q\|=1} \{\nabla\psi(x, y).Q - n(x, y)\} = 0.$$

"Lagrangian" discretisation

$$\sup_{\|Q\|\leq 1} \{\frac{\psi((x,y) - dt\, Q) - \psi(x,y)}{-dt} - n(x,y)\} = 0$$

Rem. : This is first order (if the solution is smooth !)

$$\psi((x,y) - dt\, Q) - \psi(x,y) = -dt\, \nabla\psi(x,y).Q + O(dt^2)$$

Rem. : this is upwinding

$$\psi(x,y) = \inf_{\|Q\|\leq 1} \{\psi((x,y) - dt\, Q)\} + dt\, n(x,y)$$

Recall

$$\dot{Y} = \nabla\psi = Q_{opt}\|\nabla\psi\|$$

Grid interpolation

Grid : $(x_i, y_j) = (i\,dx, j\,dy)$. Note $\psi(x_i, y_j) = \psi_{ij}$ and $n(x_i, y_j) = n_{ij}$

First assume that $(x, y) - dt\,q) \in T = \{(x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j-1})\}$ (restrict Q to point in one of the quadrant).

ψ using a "convex linear" combination of the value at grid points :

$$\psi((x, y) - dt\,q) = \alpha\,\psi_{ij} + \beta\,\psi_{i+1j} + \gamma\,\psi_{ij+1}$$

where α, β and γ are such that

$$\begin{cases} (x, y) - dt\,q = \alpha\,(x_i, y_j) + \beta\,(x_{i+1}, y_j) + \gamma\,(x_{i+1}, y_{j+1})) \\ \alpha + \beta + \gamma = 1. \end{cases}$$

Rem : Again first order approximation.

The optimization problem now depends on (α, β, γ). It can be worked out for the four quadrants and taking $dt = \frac{dx\,dy}{\sqrt{dx^2+dy^2}}$ simplifies into the discrete Hamiltonian :

$$g_{ij}(\psi_{ij}, \psi_{i+1j}, \psi_{ij+1}, \psi_{i-1j}, \psi_{ij-1}) = 0,\ \forall (i,j)$$

where

$$g_{ij}(\psi_{ij}, \psi_{i+1j}, \psi_{ij+1}, \psi_{i-1j}, \psi_{ij-1}) =$$
$$\sqrt{\max(a^+, b^-)^2 + \max(c^+, d^-)^2} - n(x_i, y_j),$$

$a^+ = \max(0, a)$, $b^- = \max(0, -b)$ and

$$a = D_x^- \psi_{ij} = \frac{\psi_{ij} - \psi_{i-1j}}{dx} \quad b = D_x^+ \psi_{ij} = \frac{\psi_{i+1j} - \psi_{ij}}{dx}$$
$$c = D_y^- \psi_{ij} = \frac{\psi_{ij} - \psi_{ij-1}}{dy} \quad d = D_y^+ \psi_{ij} = \frac{\psi_{ij+1} - \psi_{ij}}{dy}$$

Relaxation

S. J. Osher and L. Rudin
Rapid convergence of approximate solution to shape form shading problem.
Never Published

$$g_{ij}(\psi_{ij}, \psi_{i+1j}, \psi_{ij+1}, \psi_{i-1j}, \psi_{ij-1}) = 0, \forall (i,j)$$

is a system of nonlinear equations ... may be difficult to solve directly. Instead is is easy to prove that the algorithm obtained by the following relaxation scheme converges :

Compute a sequence of $(\psi^n_{ij})_{i,j}$ solutions of

$$g_{ij}(\psi^n_{ij}, \psi^{n-1}_{i+1j}, \psi^{n-1}_{ij+1}, \psi^{n-1}_{i-1j}, \psi^{n-1}_{ij-1}) = 0, \forall (i,j)$$

Note this is like reintroducing time ...

On the positive quadrant

we have to solve for $t = \psi^n_{ij}$ $(h = dx = dy)$

$$\sqrt{((\psi^{n-1}_{i+1j} - t)^-)^2 + ((\psi^{n-1}_{ij+1} - t)^-)^2} - h * n_{ij} = 0$$

Solution is explicitely given as

$$\psi^{n+1}_{ij} = 0.5\,(\psi^{n-1}_{ij+1} + \psi^{n-1}_{i+1j} + \sqrt{2h^2\,n^2_{ij} - (\psi^{n-1}_{i+1j} - \psi^{n-1}_{i+1j})^2} \quad if \quad |\psi^{n-1}_{i+1j} - \psi^{n-1}_{i+1j}| < h\,n_{ij}$$

$$\psi^{n+1}_{ij} = \min(\psi^{n-1}_{i+1j}, \psi^{n-1}_{i+1j}) + h\,n_{ij} \qquad else$$

Extend to all four quadrant by seleecting min...

Slightly faster way is to revisit as $P1$ interpolation on each triangle made by the 5 point stencil.

$$\psi((x,y) - dt\,(qx,qy)) = \psi_{ij} - dt\,q_x\,(D_x\psi_{ij}) \,-\, dt\,q_y\,(D_y\psi_{ij})$$

Three unknown $(\psi^n_{ij}, D_x\psi_{ij}, D_y\psi_{ij})$ and 2 equations $((qx,qy) = (h,0),(qx,qy) = (0,h))$

$$\psi^{n-1}_{ij+1} = \psi^n_{ij} - hD_y\psi_{ij})$$
$$\psi^{n-1}_{i+1j} = \psi^n_{ij} - hD_x\psi_{ij})$$

then close with the Eikonal equation ..

$$\|\nabla\psi(X)\|^2 = n(X)^2 \quad X = (x,y) \Leftrightarrow \sqrt{(D_x^+\psi_{ij}^+)^2 + (D_y^+\psi_{ij}^+)^2} - n_{ij} = 0$$

Extension to all four quadrant yield the scheme

Fondamental properties of the numerical (implicit!) scheme

- Consistance : For a smooth function ϕ, looking at a sequence s.t. $(i\,dx, j\,dy) \rightarrow (x, y)$ as $(dy, dx) \rightarrow 0$ and setting $\psi_{ij} = \phi(x_i, y_j)$, we get

$$\lim g_{ij}(\psi_{ij}, \psi_{i+1j}, \psi_{ij+1}, \psi_{i-1j}, \psi_{ij-1}) = \|\nabla\phi(x,y)\| - n(x,y)$$

- Monotonicity : if $(u_{ij})_{ij} < (v_{ij})_{ij}$ then for all t and all i, j we have

  $$g_{ij}(t, u_{i+1j}, u_{ij+1}, u_{i-1j}, u_{ij-1}) < g_{ij}(t, v_{i+1j}, v_{ij+1}, v_{i-1j}, v_{ij-1})$$

  (Upwind scheme acts as a "time" dependent equation in the ray direction).

- Stability : unconditionnaly stable (implicit !)

Relaxation and convergence proof

http://www.levelset.com/system/html/modules/sections/index.php?op=v

Interview with Stanley Osher at National University of Singapore :

Stanley Osher is an extraordinary mathematician who has ....

It turned out that I had a friend who knew the District Attorney or

was then doing video image enhancement with my colleague L. Rudin,

....

I: You could be rich. Hollywood would be paying you millions.

O: People work for salaries. There is money, ego and fun. It's a ve