4. Geometric optics, Rakesh's construction, and imaging and inversion in the presence of multipathing.

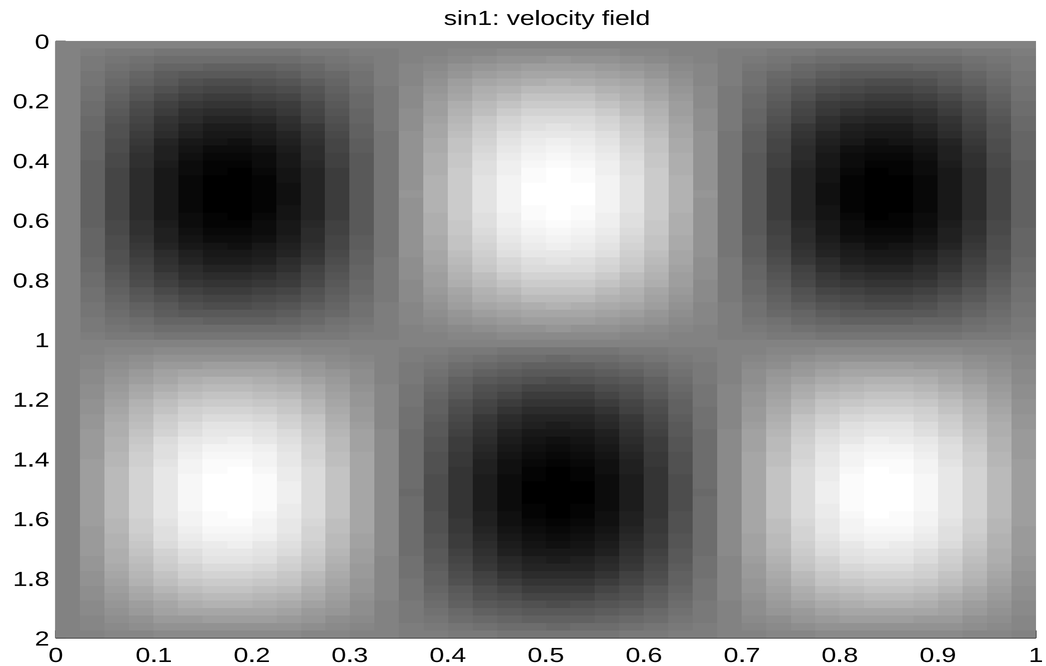
The theory developed by Beylkin and others cannot be the end of the story:

- The "single ray" hypotheses generally fails in the presence of strong refraction.

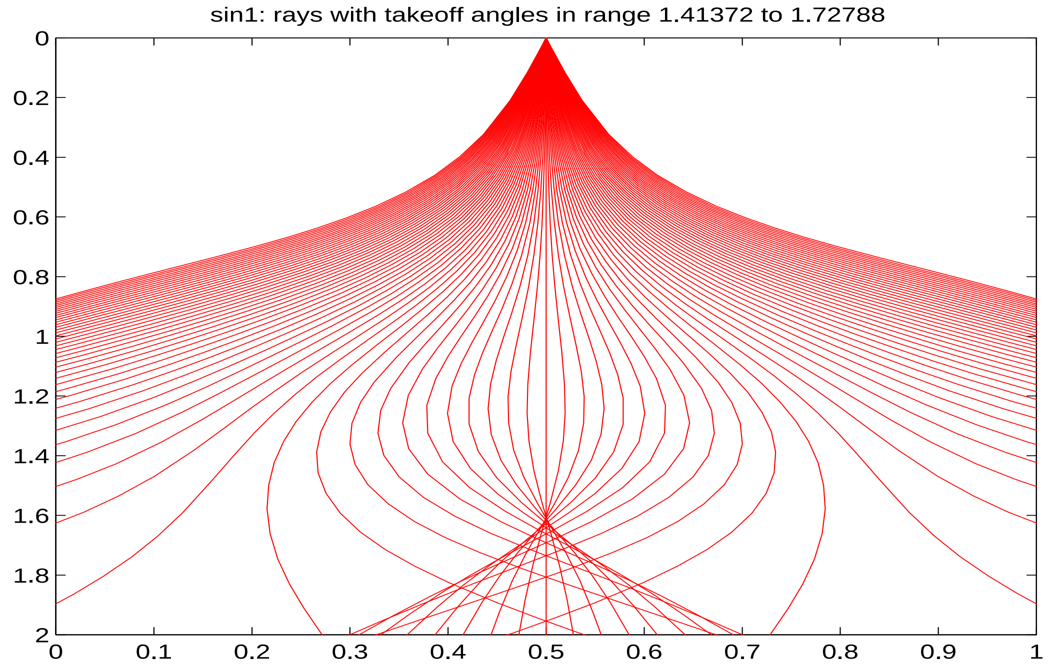
- B. White, "The Stochastic Caustic" (1982): For "random but smooth" $v(\mathbf{x})$ with variance σ, points at distance $O(\sigma^{-2/3})$ from source have more than one ray connecting to source, with high probability 1 - *multipathing* associated with formation of *caustics* = ray envelopes.

- Formation of caustics invalidates asymptotic analysis on which Beylkin result is based.

- Strong refraction leading to multipathing and caustic formation typical of salt (4-5 km/s) intrusion into sedimentary rock (2-3 km/s) (eg. Gulf of Mexico), also chalk tectonics in North Sea and elsewhere - some of the most promising petroleum provinces!



2D Example of strong refraction: Sinusoidal velocity field $v(x,z) = 1 + 0.2\sin\frac{\pi z}{2}\sin 3\pi x$



Rays in sinusoidal velocity field, source point = origin. Note formation of caustic, multiple rays to source point in lower center.

How do we get away from “simple geometric optics”, SSR, D-SR,... - all violated in sufficiently complex (and realistic) models?

Rakesh *Comm. PDE* 1988, Nolan *Comm. PDE* 1997: global description of $F_\delta[v]$ as mapping reflectors $\mapsto$ reflections.

$Y = \{\mathbf{x}_s, t, \mathbf{x}_r\}$ (time $\times$ set of source-receiver pairs) submfd of $\mathbf{R}^7$ of dim. ≤ 5, $\Pi : T^*(\mathbf{R}^7) \to T^*Y$ the natural projection

$\mathrm{supp}\, r \subset X \subset \mathbf{R}^3$

Canonical relation $C_{F_\delta[v]} \subset T^*(X) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$ describes singularity mapping properties of F:

$$(\mathbf{x}, \xi, \mathbf{y}, \eta) \in C_{F_\delta[v]} \Leftrightarrow$$

$$\text{for some}\, u \in \mathcal{E}'(X),\ (\mathbf{x}, \xi) \in WF(u),\ \text{and}\ (\mathbf{y}, \eta) \in WF(Fu)$$

Rays of geometric optics: solutions of Hamiltonian system

$$\frac{d\mathbf{X}}{dt} = \nabla_{\mathbf{\Xi}} H(\mathbf{X}, \mathbf{\Xi}), \ \frac{d\mathbf{\Xi}}{dt} = -\nabla_{\mathbf{X}} H(\mathbf{X}, \mathbf{\Xi})$$

with $H(\mathbf{X}, \mathbf{\Xi}) = 1 - v^2(\mathbf{X})|\mathbf{\Xi}|^2 = 0$ (*null bicharacteristics*).

Characterization of C_F:

$$((\mathbf{x}, \xi), (\mathbf{x}_s, t, \mathbf{x}_r, \xi_{\mathbf{s}}, \tau, \xi_{\mathbf{r}})) \in C_{F_\delta[v]} \subset T^*(X) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$$

$\Leftrightarrow$ there are *rays of geometric optics* $(\mathbf{X}_s, \mathbf{\Xi}_s)$, $(\mathbf{X}_r, \mathbf{\Xi}_r)$ and times t_s, t_r so that

$$\Pi(\mathbf{X}_s(0), t, \mathbf{X}_r(t), \mathbf{\Xi}_s(0), \tau, \mathbf{\Xi}_r(t)) = (\mathbf{x}_s, t, \mathbf{x}_r, \xi_s, \tau, \xi_r),$$

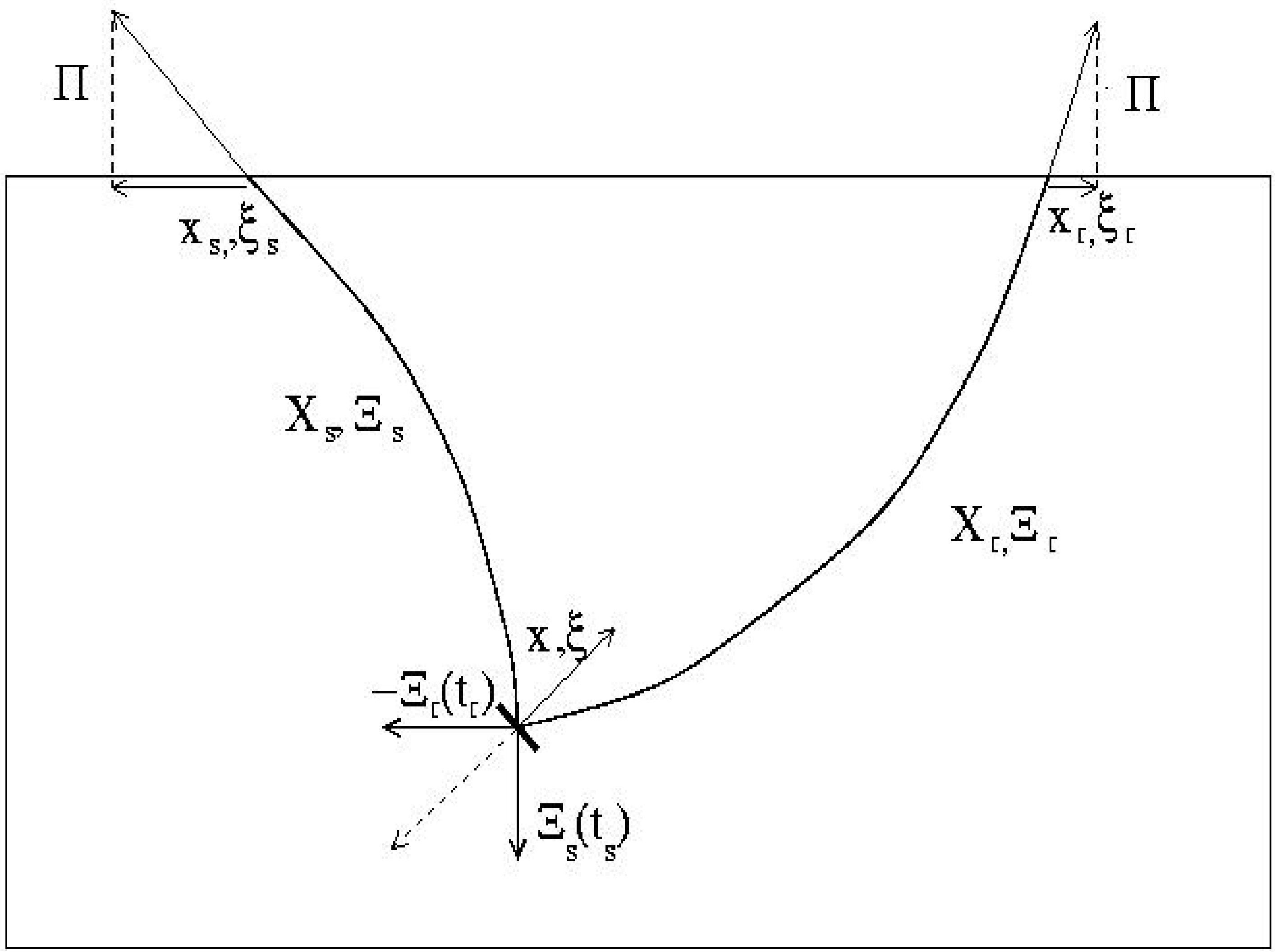
$$\mathbf{X}_s(t_s) = \mathbf{X}_r(t - t_r) = \mathbf{x}, \ t_s + t_r = t, \ \mathbf{\Xi}_s(t_s) - \mathbf{\Xi}_r(t - t_r) || \xi$$

Since $\Xi_s(t_s)$, $-\Xi_r(t-t_r)$ have same length, sum = bisector $\Rightarrow$ *velocity vectors of incident ray from source and reflected ray from receiver (traced backwards in time) make equal angles with reflector at* $\mathbf{x}$ *with normal* ξ.

Upshot: canonical relation of $F_\delta[v]$ simply enforces the equal-angles law of reflection.

Further, *rays carry high-frequency energy*, in exactly the fashion that seismologists imagine.

Finally, *Rakesh's characterization of* C_F *is global:* no assumptions about ray geometry, other than no forward scattering and no grazing incidence on the acquisition surface Y, are needed.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Plan of attack: recall that

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial \delta u}{\partial t}(\mathbf{x}_r, t; \mathbf{x}_s)$$

where

$$\frac{1}{v^2}\frac{\partial^2 \delta u}{\partial t^2} - \nabla^2 \delta u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} r$$

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = \delta(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

and $u, \delta u \equiv 0$, $t < 0$.

Need to understand (1) $WF(u)$, (2) relation $WF(r) \leftrightarrow WF(ru)$, (3) WF of soln of WE in terms of WF of RHS (this also gives (1)!).

(1) Singularities of u

Main tool: **Propagation of Singularities** theorem of Hörmander (1970).

Given symbol $p(\mathbf{x}, \xi)$, order m, with asymptotic expansion, define *null bicharateristics* (= rays) as solutions $(\mathbf{x}(t), \xi(t))$ of Hamiltonian system

$$\frac{d\mathbf{x}}{dt} = \frac{\partial p}{\partial \xi}(\mathbf{x}, \xi), \ \frac{d\xi}{dt} = -\frac{\partial p}{\partial \mathbf{x}}(\mathbf{x}, \xi)$$

with $p(\mathbf{x}(t), \xi(t)) \equiv 0$.

Theorem: Suppose $p(\mathbf{x}, D)u = f$, and suppose that for $t_0 \leq t \leq t_1$, $(\mathbf{x}(t), \xi(t)) \notin WF(f)$. Then either $\{(\mathbf{x}(t), \xi(t)) : t_0 \leq t \leq t_1\} \subset WF(u)$ or $\{(\mathbf{x}(t), \xi(t)) : t_0 \leq t \leq t_1\} \subset T^*(\mathbf{R}^n) - WF(u)$.

RHS of wave equation for $u = \delta$ function in $\mathbf{x}, t$. WF set $= \{(\mathbf{x}, t, \xi, \tau) : \mathbf{x} = \mathbf{x}_s, t = 0\}$ - i.e. no restriction on covector part.

$\Rightarrow (\mathbf{x}, t, \xi, \tau) \in WF(u)$ iff a ray starting at $(\mathbf{x}_s, 0)$ passes over $(\mathbf{x}, t)$ - i.e. $(\mathbf{x}, t)$ lies on the "light cone" with vertex at $(\mathbf{x}_x, 0)$. Symbol for wave op is $p(\mathbf{x}, t, \xi, \tau) = \frac{1}{2}(\tau^2 - v^2(\mathbf{x})|\xi|^2)$, so Hamilton's equations for null bicharacteristics are

$$\frac{d\mathbf{X}}{dt} = -v^2(\mathbf{X})\mathbf{\Xi},\ \frac{d\mathbf{\Xi}}{dt} = \nabla \log v(\mathbf{X})$$

Thus ξ is proportional to velocity vector of ray.

[(ξ, τ) *normal* to light cone.]

(2) Wavefront set of ru (Gabor calculus: Duistermaat, Ch. 1)

Here r is really $(r \circ \pi)u$, where $\pi(\mathbf{x},t) = \mathbf{x}$. Choose bump function ϕ localized near $(\mathbf{x},t)$

$$\phi(\widehat{r \circ \pi)u}(\xi,\tau) = \int d\xi'\, d\tau' \widehat{\phi r}(\xi')\delta(\tau')\hat{u}(\xi-\xi',\tau-\tau')$$

$$= \int d\xi' \widehat{\phi r}(\xi')\hat{u}(\xi-\xi',\tau)$$

This will decay rapidly as $|(\xi,\tau)| \to \infty$ unless (i) you can find $(\mathbf{x}',\xi') \in WF(r)$ so that $\mathbf{x},\mathbf{x}' \in \pi(\text{supp}\phi)$, $\xi - \xi' \in WF(u)$, i.e. $(\xi,\tau) \in WF(r\circ\pi) + WF(u)$, or (ii) $\xi \in WF(r)$ or $(\xi,\tau) \in WF(u)$.

Possibility (ii) will not contribute, so effectively

$$WF((r\circ\pi)u) = \{(\mathbf{x},t_s,\xi+\Xi_s(t_s),\cdot) : (\mathbf{x},\xi) \in WF(r),\, \mathbf{x} = \mathbf{X}_s(t_s)$$

for a ray $(\mathbf{X}_s,\Xi_s)$ with $\mathbf{X}_s(0) = x_s$, some τ.

(3) Wavefront set of δu

Once again use propagation of singularities: $(\mathbf{x}_r, t, \xi_r, \tau_r) \in WF(\delta u) \Leftrightarrow$ on ray $(\mathbf{X}_r, \mathbf{\Xi}_r)$ passing through $WF(ru)$. Can argue that time of intersection is $t - t_r < t$.

That is,

$$\mathbf{X}_r(t) = \mathbf{x}_r, \mathbf{X}_r(t - t_r) = \mathbf{X}_s(t_s) = x,$$

$t = t_r + t_s$, and

$$\mathbf{\Xi}_r(t_s) = \xi + \mathbf{\Xi}_s(t_s)$$

for some $\xi \in WF(r)$. **Q. E. D.**

Rakesh also showed that $F[v]$ is a *Fourier Integral Operator* = class of oscillatory integral operators, introduced by Hörmander and others in the '70s to describe the solutions of nonelliptic PDEs.

Phases and amplitudes of FIOs satisfy certain restrictive conditions. Canonical relations have geometric description similar to that of $F[v]$. Adjoint of FIO is FIO with inverse canonical relation.

ΨDOs are special FIOs.

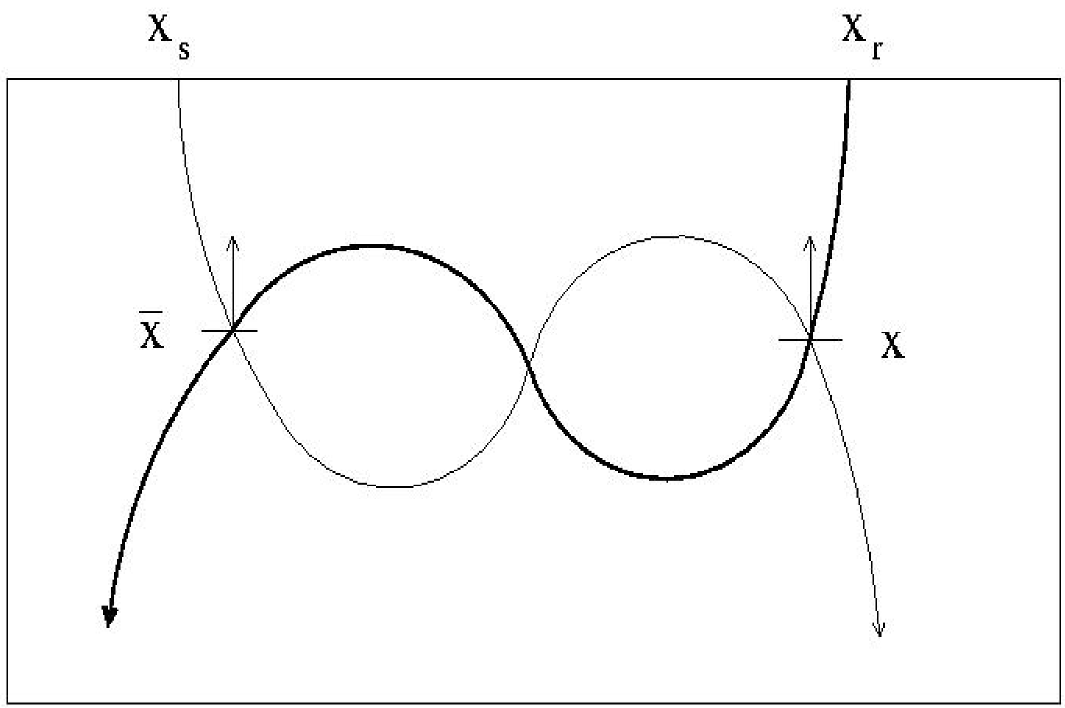
Composition of FIOs does *not* yield an FIO in general. Beylkin had shown that $F[v]^*F[v]$ is FIO (ΨDO, actually) under simple ray geometry hypothesis - but this is only sufficient.

Smit, tenKroode and Verdel (1998): provided that

- source, receiver positions $(\mathbf{x}_s, \mathbf{x}_r)$ form an *open* 4D manifold ("complete coverage" - all source, receiver positions at least locally), and
- the *Traveltime Injectivity Condition* ("TIC") holds: $C_{F[v]}^{-1} \subset T^*Y - \{0\} \times T^*X - \{0\}$ is a *function* - that is, initial data for source and receiver rays and total travel time together determine reflector uniquely.

then $F[v]^*F[v]$ is ΨDO $\Rightarrow$ application of $F[v]^*$ produces image, and $F[v]^*F[v]$ has microlocal parametrix ("asymptotic inversion").

TIC is nontrivial constraint:



Symmetric waveguide: time ($\mathbf{x}_s \to \bar{\mathbf{x}} \to \mathbf{x}_r$) same as time ($\mathbf{x}_s \to \mathbf{x} \to \mathbf{x}_r$), so TIC fails.

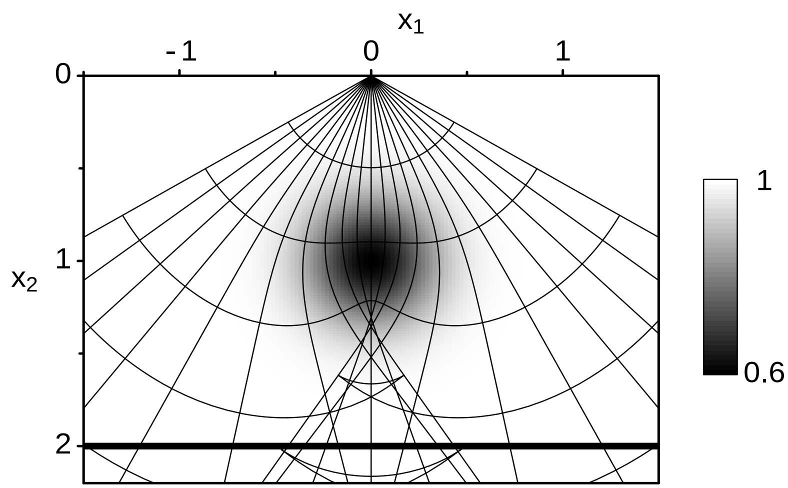
Stolk (2000): under "complete coverage" hypothesis, v for which $F[v]^*F[v]$ is = [ΨDO + rel. smoothing op] form open, dense set.

NB: application of $F[v]^*$ involves accounting for *all* rays connecting source and receiver with reflectors. Standard practice still attempts imaging with single choice of ray pair (shortest time, max energy,...). Operto et al (2000) give nice illustration that all rays must be included.

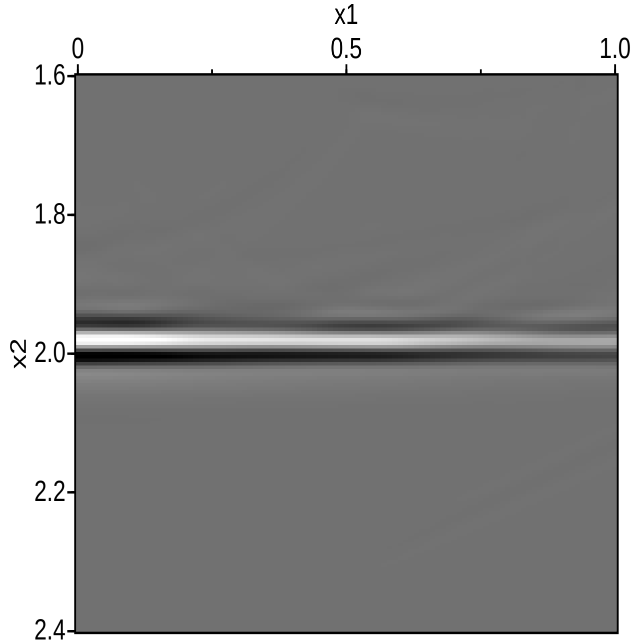
Limitation of Smit-tenKroode-Verdel: most idealized data acquisition geometries violate "complete coverage": for example, idealized marine streamer geometry (src-recvr submfd is 3D)

Nolan (1997): result remains true without "complete coverage" condition: requires only TIC plus addl condition so that projection $C_{F[v]} \to T^*Y$ is embedding - but examples violating TIC are much easier to construct when source-receiver submfd has positive codim.

Synthetic 2D Example (Stolk and WWS, 2001): Strongly refracting acoustic lens (v) over horizontal reflector (r), $S^{\text{obs}} = F[v]r$. (i) for open source-receiver set, $F[v]^*S^{\text{obs}}$ = good image of reflector - within limits of finite frequency implied by numerical method, $F[v]^*F[v]$ acts like ΨDO; (ii) for *common offset* submfd (codim 1), TIC is violated and $WF(F[v]^*S^{\text{obs}})$ is larger than $WF(r)$.



Example (Stolk & WWS, 2001): Gaussian lens over flat reflector at depth 2 km ($r(\mathbf{x}) = \delta(x_1 - 2)$, $x_1 =$ depth).



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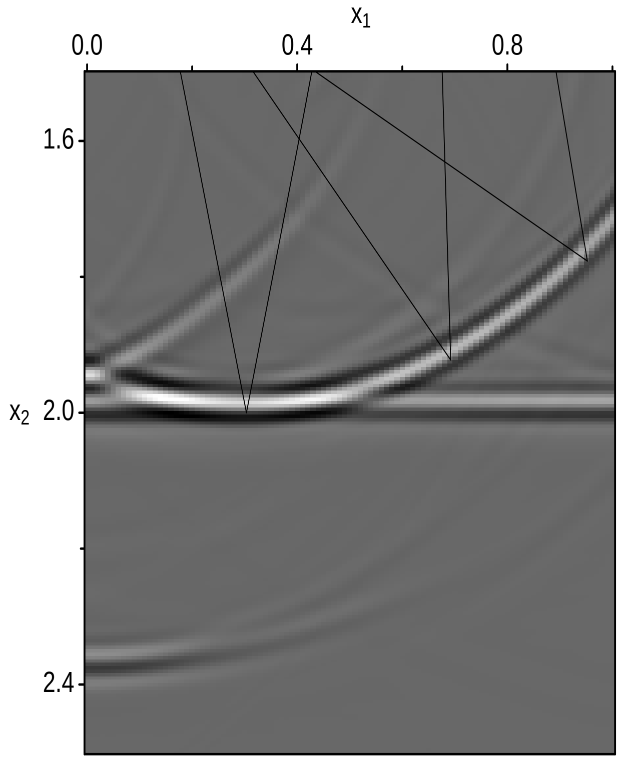
$F[v]^* S^{\text{obs}}$ for complete coverage (all source and receiver positions): good image of reflector.



Same, but for single offset - codim 1: TIC is violated, $F[v]^*F[v]$ is not ΨDO. Image overlain with ray pairs sharing same first factor in $C_{F[v]}^{-1}$ (i.e. these reflect at 3 reflecting elements corresponding to same reflection).