5. A step beyond linearization: velocity analysis

Partially linearized seismic inverse problem ("velocity analysis"): given observed seismic data S^{obs}, find smooth *velocity* $v \in \mathcal{E}(X), X \subset \mathbf{R}^3$ oscillatory *reflectivity* $r \in \mathcal{E}'(X)$ so that

$$F[v]r \simeq S^{\text{obs}}$$

Acoustic partially linearized model: acoustic potential field u and its perturbation δu solve

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u = \delta(t)\delta(\mathbf{x} - \mathbf{x}_s), \;\; \left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta u = 2r\nabla^2 u$$

plus suitable bdry and initial conditions.

$$F[v]r = \left.\frac{\partial \delta u}{\partial t}\right|_Y$$

data acquisition manifold $Y = \{(\mathbf{x}_r, t; \mathbf{x}_s)\} \subset \mathbf{R}^7$, dimn $Y \leq 5$ (many idealizations here!).

$F[v] : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ is a linear map (FIO of order 1), but dependence on v is quite nonlinear, so this inverse problem is nonlinear.

Agenda:

- reformulation of inverse problem via *extensions*
- "standard processing" extension and standard VA
- the surface oriented extension and standard MVA
- the ΨDO property and why it's important
- global failure of the ΨDO property for the SOE
- Claerbout's depth oriented extension has the ΨDO property

Extension of $F[v]$: manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

$$\begin{array}{ccccc} & & \bar{F}[v] & & \\ & \mathcal{E}'(\bar{X}) & \rightarrow & \mathcal{D}'(Y) & \\ \chi & \uparrow & & \uparrow & \text{id} \\ & \mathcal{E}'(X) & \rightarrow & \mathcal{D}'(Y) & \\ & & F[v] & & \end{array}$$

commutes.

Invertible extension: $\bar{F}[v]$ has a *right parametrix* $\bar{G}[v]$, i.e. $I - \bar{F}[v]\bar{G}[v]$ is smoothing. [The trivial extension - $\bar{X} = X, \bar{F} = F$ - is virtually never invertible.] Also χ has a *left inverse* η.

Reformulation of inverse problem: given S^{obs}, find v so that $\bar{G}[v]S^{\text{obs}} \in \mathcal{R}(\chi)$ (implicitly determines r also!).

Example 1: Standard VA extension. Treat each CMP *as if it were the result of an experiment performed over a layered medium*, but permit the layers to vary with midpoint.

Thus $v = v(z), r = r(z)$ for purposes of analysis, but at the end $v = v(\mathbf{x}_m, z), r = r(\mathbf{x}_m, z)$.

$$F[v]R(\mathbf{x}_m, h, t) \simeq A(\mathbf{x}_m, h, z(\mathbf{x}_m, h, t))R(\mathbf{x}_m, z(\mathbf{x}_m, h, t))$$

Here $z(\mathbf{x}_m, h, t)$ is the inverse of the 2-way traveltime

$$t(\mathbf{x}_m, h, z) = 2\tau(\mathbf{x}_m + (h, 0, z), \mathbf{x}_m)_{v=v(\mathbf{x}_m, z)}$$

computed with the layered velocity $v(\mathbf{x}_m, z)$, i.e. $z(\mathbf{x}_m, h, t(\mathbf{x}_m, h, z')) = z'$.

R is (yet another version of) "reflectivity"

$$R(\mathbf{x}_m, z) = \frac{1}{2}\frac{dr}{dz}(\mathbf{x}_m, z)$$

That is, $F[v]$ *is a change of variable followed by multiplication by a smooth function*. NB: industry standard practice is to use vertical traveltime t_0 instead of z for depth variable.

Can write this as $F[v] = \bar{F}S^*$, where $\bar{F}[v] = N[v]^{-1}M[v]$ has right parametrix $\bar{G}[v] = M[v]N[v]$:

$N[v]$ = **NMO operator** $N[v]d(\mathbf{x}_m, h, z) = d(\mathbf{x}_m, h, t(\mathbf{x}_m, h, z))$

$M[v]$ = multiplication by A

S = **stacking operator**

$$Sf(\mathbf{x}_m, z) = \int dh\, f(\mathbf{x}_m, h, z),\;\; S^*r(\mathbf{x}_m, h, z) = r(\mathbf{x}, z)$$

Identify as extension: $\bar{F}[v], \bar{G}[v]$ as above, $X = \{\mathbf{x}_m, z\}, H = \{h\}, \bar{X} = X \times H, \chi = S^*, \eta = S$ - the invertible extension properties are clear.

Standard names for the Standard VA extension objects: $\bar{F}[v]$ = "inverse NMO", $\bar{G}[v]$ = "NMO" [often the multiplication op $M[v]$ is neglected]; η = "stack", χ = "spread"

How this is used for velocity analysis: Look for v that makes $\bar{G}[v]d \in \mathcal{R}(\chi)$

So what is $\mathcal{R}(\chi)$? $\chi[r](\mathbf{x}_m, z, h) = r(\mathbf{x}_m, z)$ Anything in range of χ is *independent of* h. Practical issues $\Rightarrow$ replace "independent of" with "smooth in".

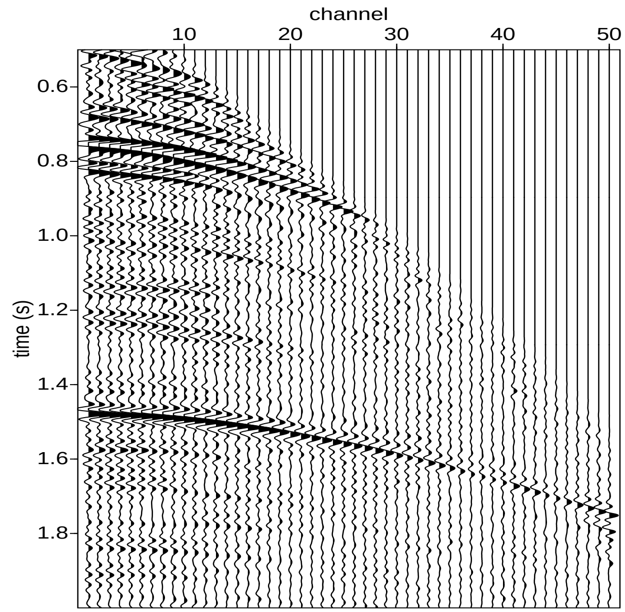
Inverse problem reduced to: adjust v to make $\bar{G}[v]d^{\mathsf{obs}}$ smooth in h, i.e. *flat* in z, h display for each $\mathbf{x}_m$ (*NMO-corrected CMP*).

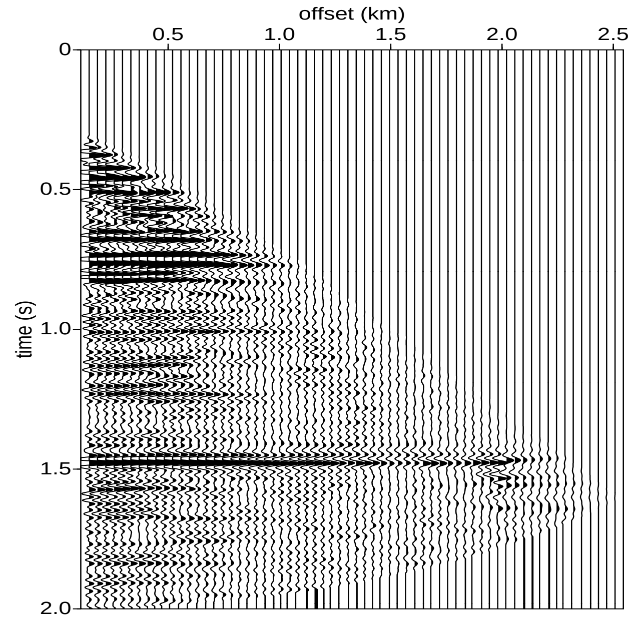
Replace z with t_0, v with v_{RMS} em localizes computation: reflection through $\mathbf{x}_m, t_0, 0$ *flattened* by adjusting $v_{\mathsf{RMS}}(\mathbf{x}_m, t_0) \Rightarrow$ 1D search, do by visual inspection.

Various aids - NMO corrected CMP gathers, velocity spectra, etc.

See: Claerbout: *Imaging the Earth's Interior*

WWS: MGSS 2000 notes





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Left: part of survey (S^{obs}) from North Sea (thanks: Shell Research), lightly preprocessed.
Right: restriction of $\bar{G}[v]S^{\mathrm{obs}}$ to $\mathbf{x}_m = \text{const}$ (function of depth, offset): shows rel. sm'ness in h (offset) for properly chosen v.

This only works where Earth is "nearly layered". Where this fails, go to **Example 2: Surface oriented or standard MVA extension**.

Shot version: $\Sigma_s =$ set of shot locations, $\bar{X} = X \times \Sigma_s$, $\chi[r](\mathbf{x}, \mathbf{x}_s) = r(\mathbf{x})$.

$$\bar{F}[v]\bar{r}(\mathbf{x}_r, t, \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int dx\, \bar{r}(\mathbf{x}, \mathbf{x}_s) \int ds\, G(\mathbf{x}_r, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{x})$$

Offset version (preferred because it minimizes truncation artifacts): $\Sigma_h =$ set of half-offsets in data, $\bar{X} = X \times \Sigma_h$, $\chi[r](\mathbf{x}, \mathbf{h}) = r(\mathbf{x})$.

$$\bar{F}[v]\bar{r}(\mathbf{x}_s, t, \mathbf{h}) = \frac{\partial^2}{\partial t^2} \int dx\, \bar{r}(\mathbf{x}, \mathbf{h}) \int ds\, G(\mathbf{x}_s + \mathbf{h}, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{x})$$

[Parametrize data with source location $\mathbf{x}_s$, time t, offset $\mathbf{h}$.] **NB:** note that both versions are "block diagonal" - family of operators (FIOs - tenKroode lectures) parametrized by $\mathbf{x}_s$ or $\mathbf{h}$.

Properties of surface oriented extension (Beylkin (1985), Rakesh (1988)): if $\|v\|_{C^2(X)}$ "not too big", then

- $\bar{F}$ has **the ΨDO property**: $\bar{F}\bar{F}^*$ is ΨDO

- singularities of $\bar{F}\bar{F}^* d \subset$ singularities of d

- straightforward construction of right parametrix $\bar{G} = \bar{F}^* Q$, $Q =$ ΨDO, also as generalized Radon Transform - explicitly computable.

Range of χ (offset version): $\bar{r}(\mathbf{x}, \mathbf{h})$ independent of $\mathbf{h} \Rightarrow$ "semblance principle": find v so that $\bar{G}[v]d^{\text{obs}}$ is independent of $\mathbf{h}$. Practical limitations $\Rightarrow$ replace "independent of $\mathbf{h}$" by "smooth in $\mathbf{h}$".

Application of these ideas = industrial practice of migration velocity analysis.

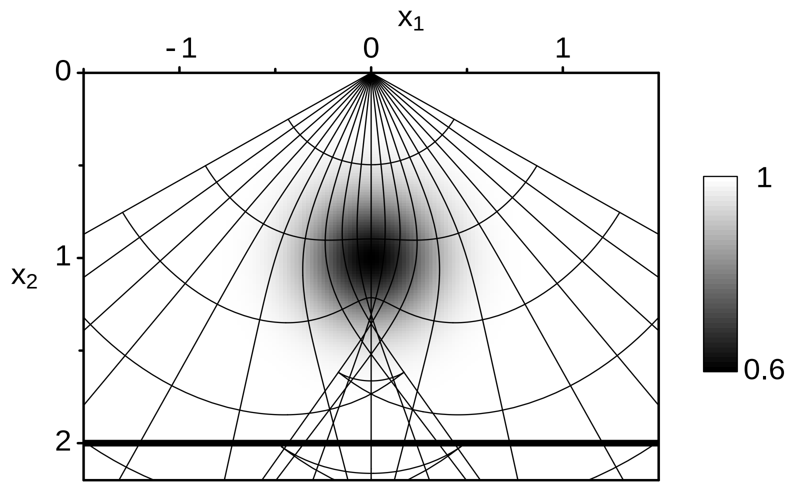
Idea: twiddle v until $\bar{G}[v]d^{\mathrm{obs}}$ is smooth in $\mathbf{h}$.

Since it is hard to inspect $\bar{G}[v]d^{\mathrm{obs}}(x,y,z,h)$, pull out subset for constant x, y = **common image gather** ("CIG"): display function of z, h for fixed x, y. These play same role as NMO corrected CMP gathers in layered case.

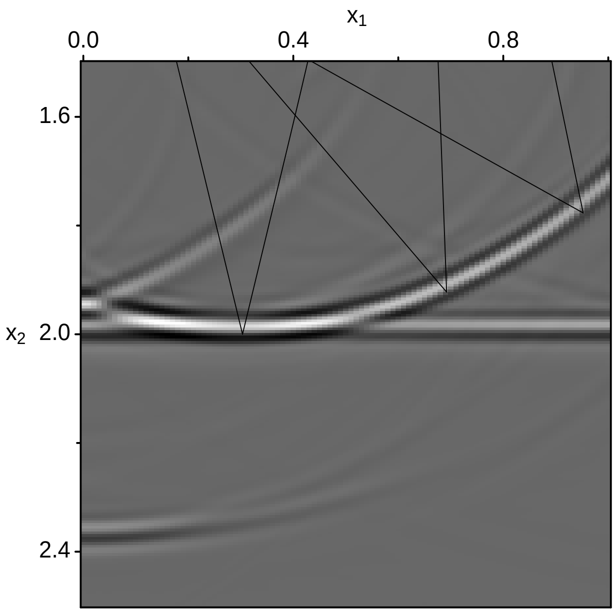
Try to adjust v so that selected CIGs are *flat* - just as in Standard VA. This is much harder, as there is no RMS velocity trick to localize the computation - each CIG depends globally on v.

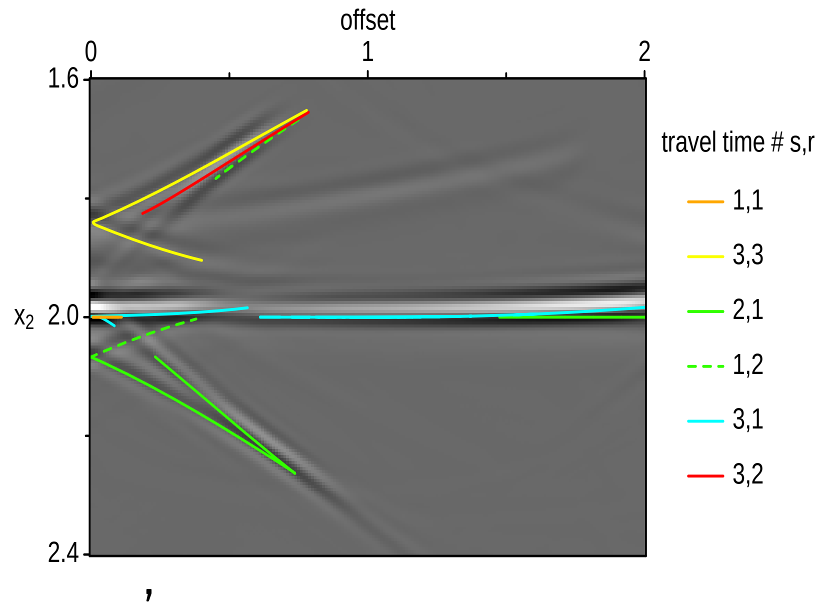
Description, some examples: Yilmaz, *Seismic Data Processing*.

Nolan (1997): big trouble! In general, standard extension does **not** have the ΨDO property. Geometric optics analysis: for $\|v\|_{C^2(X)}$ "large", multiple rays connect source, receiver to reflecting points in X; block diagonal structure of $\bar{F}[v] \Rightarrow$ info necessary to distinguish multiple rays is *projected out*.



Example (Stolk & WWS, 2001): Gaussian lens over flat reflector at depth z ($r(\mathbf{x}) = \delta(x_1 - z)$, $x_1 =$ depth).





Left: Const. h slice of $\bar{G}d^{\text{obs}}$: several refl. points corresponding to same singularity in d^{obs}.
Right: CIG (const. x, y slice) of $\bar{G}d^{\text{obs}}$: not smooth in h!

Standard MVA extension only works when Earth has simple ray geometry. When this fails, go to

Example 3: Claerbout's depth oriented extension.

Σ_d = somewhat arbitrary set of vectors near 0 ("offsets"), $\bar{X} = X \times \Sigma_d$, $\chi[r](\mathbf{x},\mathbf{h}) = r(\mathbf{x})\delta(\mathbf{h})$, $\eta[\bar{r}](\mathbf{x}) = \bar{r}(\mathbf{x},0)$

$$\bar{F}[v]\bar{r}(\mathbf{x}_s,t,\mathbf{x}_r) = \frac{\partial^2}{\partial t^2}\int dx \int_{\Sigma_d} dh\, \bar{r}(\mathbf{x},\mathbf{h}) \int ds\, G(\mathbf{x}_s,t-s;\mathbf{x}+2\mathbf{h})G(\mathbf{x}_r,s;\mathbf{x})$$

$$= \frac{\partial^2}{\partial t^2}\int dx \int_{\mathbf{x}+2\Sigma_d} dy\, \bar{r}(\mathbf{x},\mathbf{y}-\mathbf{x}) \int ds\, G(\mathbf{x}_s,t-s;\mathbf{y})G(\mathbf{x}_r,s;\mathbf{x})$$

NB: in this formulation, there appears to be too many model parameters.

Computationally economical: for each $\mathbf{x}_s$ solve

$$\bar{F}[v]\bar{r}(\mathbf{x}_r, t; \mathbf{x}_s) = u(\mathbf{x}, t; \mathbf{x}_s)|_{\mathbf{x}=\mathbf{x}_r}$$

where

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) u(\mathbf{x}, t; \mathbf{x}_s) = \int_{\mathbf{x}+2\Sigma_d} dy\, \bar{r}(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, t; \mathbf{x}_s)$$

$$\left(\frac{1}{v(\mathbf{y})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{y}}\right) G(\mathbf{y}, t; \mathbf{x}_s) = \delta(t)\delta(\mathbf{x}_s - \mathbf{y})$$

Finite difference scheme: form RHS for eqn 1, step u forward in t, step G forward in t.

Computing $\bar{G}[v]$: instead of parametrix, be satisfied with adjoint.

Reverse time adjoint computation - specify adjoint field as in standard reverse time prestack migration:

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) w(\mathbf{x}, t; \mathbf{x}_s) = \int dx_r \, d(\mathbf{x}_r, t; \mathbf{x}_s)\delta(\mathbf{x} - \mathbf{x}_r)$$

with $w(\mathbf{x}, t; \mathbf{x}_s) = 0, t >> 0$.

Then

$$\bar{F}[v]^* d(\mathbf{x}, \mathbf{h}) = \int dx_s \int dt \, G(\mathbf{x} + 2\mathbf{h}, t; \mathbf{x}_s) w(\mathbf{x}, t; \mathbf{x}_s)$$

i.e. exactly the same computation as for reverse time prestack, except that crosscorrelation occurs at an offset $2\mathbf{h}$.

What should be the character of the image when the velocity is correct?

Hint: for simulation of seismograms, the input reflectivity had the form $r(\mathbf{x})\delta(\mathbf{h})$.

Therefore guess that when velocity is correct, *image is concentrated near* $h = 0$.

Examples: 2D finite difference implementation of reverse time method. Correct velocity $\equiv 1$. Input reflectivity used to generate synthetic data: random! For output reflectivity (image of $\bar{F}[v]^*$), constrain offset to be horizontal: $\bar{r}(\mathbf{x},\mathbf{h}) = \tilde{r}(\mathbf{x},h_1)\delta(h_3)$. Display CIGs (i.e. $x_1 =$const. slices).

Stolk and deHoop, 2001: Claerbout extension has the ΨDO property, at least when restricted to $\bar{r}$ of the form $\bar{r}(\mathbf{x},\mathbf{h}) = R(\mathbf{x},h_1,h_2)\delta(h_3)$, and under DSR assumption.

Sketch of proof (after Rakesh, 1988):

This will follow from *injectivity* of wavefront or *canonical relation* $C_{\bar{F}} \subset T^*(\bar{X}) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$ which describes singularity mapping properties of $\bar{F}$:

$$(\mathbf{x},\mathbf{h},\xi,\nu,\mathbf{y},\eta) \in C_{F_\delta[v]} \Leftrightarrow$$

$$\text{for some}\, u \in \mathcal{E}'(\bar{X}),\ (\mathbf{x},\mathbf{h},\xi,\nu) \in WF(u),\ \text{and}\ (\mathbf{y},\eta) \in WF(\bar{F}u)$$

Characterization of $C_{\bar{F}}$:

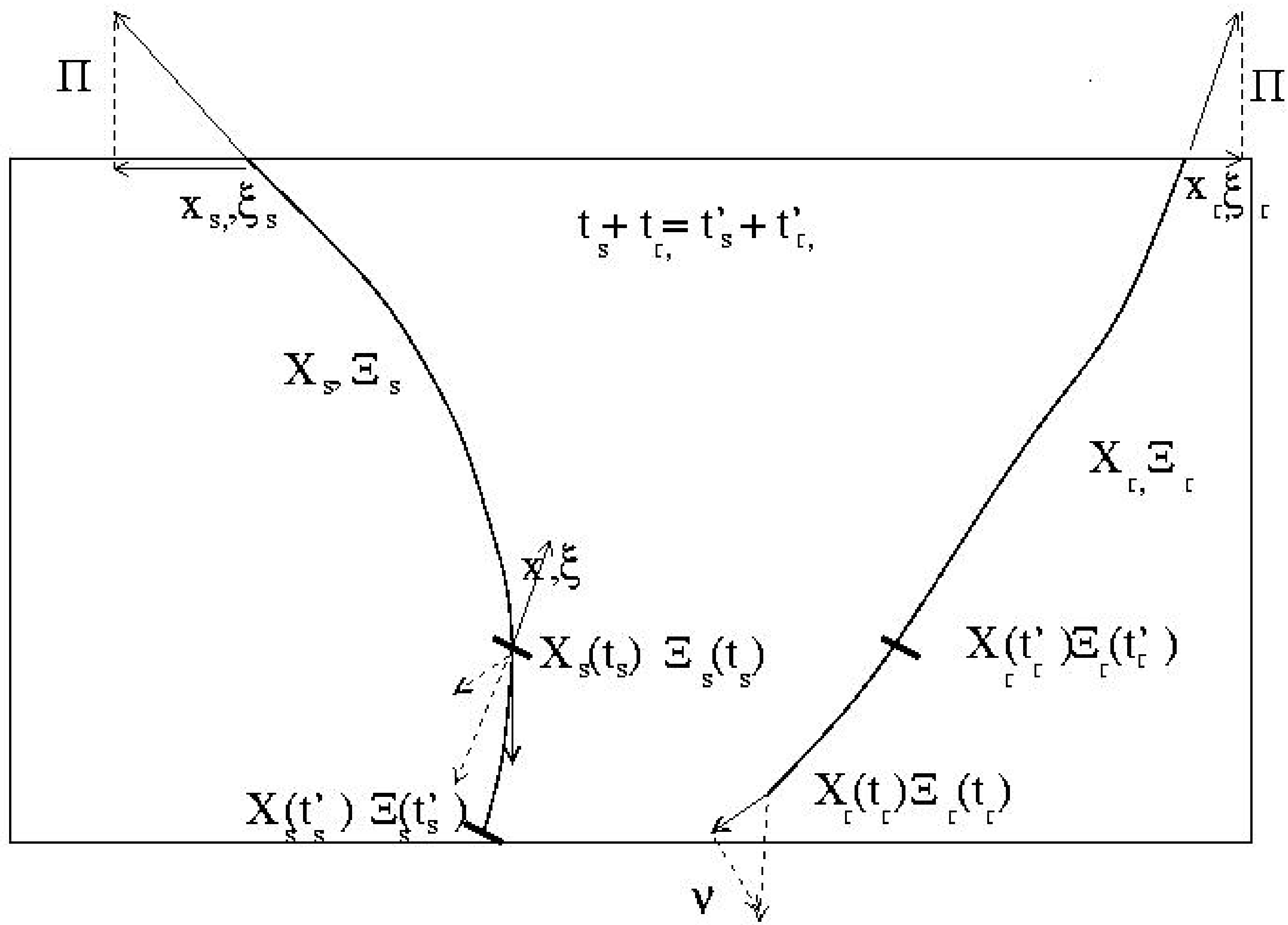
$((\mathbf{x},\mathbf{h},\xi,\nu),(\mathbf{x}_s,t,\mathbf{x}_r,\xi_{\mathbf{s}},\tau,\xi_{\mathbf{r}})) \in C_{\bar{F}}[v] \subset T^*(\bar{X}) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$

$\Leftrightarrow$ there are *rays of geometric optics* $(\mathbf{X}_s,\Xi_s)$, $(\mathbf{X}_r,\Xi_r)$ and times t_s,t_r so that

$$\Pi(\mathbf{X}_s(0),t,\mathbf{X}_r(0),\Xi_s(0),\tau,\Xi_r(0)) = (\mathbf{x}_s,t,\mathbf{x}_r,\xi_s,\tau,\xi_r),$$

$$\mathbf{X}_s(t_s) = \mathbf{x}, \mathbf{X}_r(t_r) = \mathbf{x}+2\mathbf{h},\ t_s+t_r = t,$$

$$\Xi_s(t_s)+\Xi_r(t_r)||\xi, \Xi_s(t_s)-\Xi_r(t_r)||\nu$$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Proof: uses wave equations for u, G and

- Gabor calculus: computes wave front sets of products, pull-backs, integrals, etc. See Duistermaat, Ch. 1.

- Propagation of Singularities Theorem

and that's all! [No integral representations, phase functions,...]

Note intrinsic ambiguity: if you have a ray pair, move times t_s, t_r resp. t'_s, t'_r, for which $t_s + t_r = t'_s + t'_r = t$ then you can construct two points $(\mathbf{x}, \mathbf{h}, \xi, \nu), (\mathbf{x}', \mathbf{h}', \xi', \nu')$ which are candidates for membership in $WF(\bar{r})$ and which satisfy the above relations with the same point in the cotangent bundle of $T^*(Y)$.

No wonder - there are too many model parameters!

Stolk and deHoop fix this ambiguity by imposing two constraints:

- DSR assumption: all rays carrying significant reflected energy (source or receiver) are upcoming.
- Restrict $\bar{F}$ to the domain $\mathcal{Z} \subset \mathcal{E}'(\bar{X})$

$$\bar{r} \in \mathcal{Z} \Leftrightarrow \bar{r}(\mathbf{x}, \mathbf{h}) = R(\mathbf{x}, h_1, h_2)\delta(h_3)$$

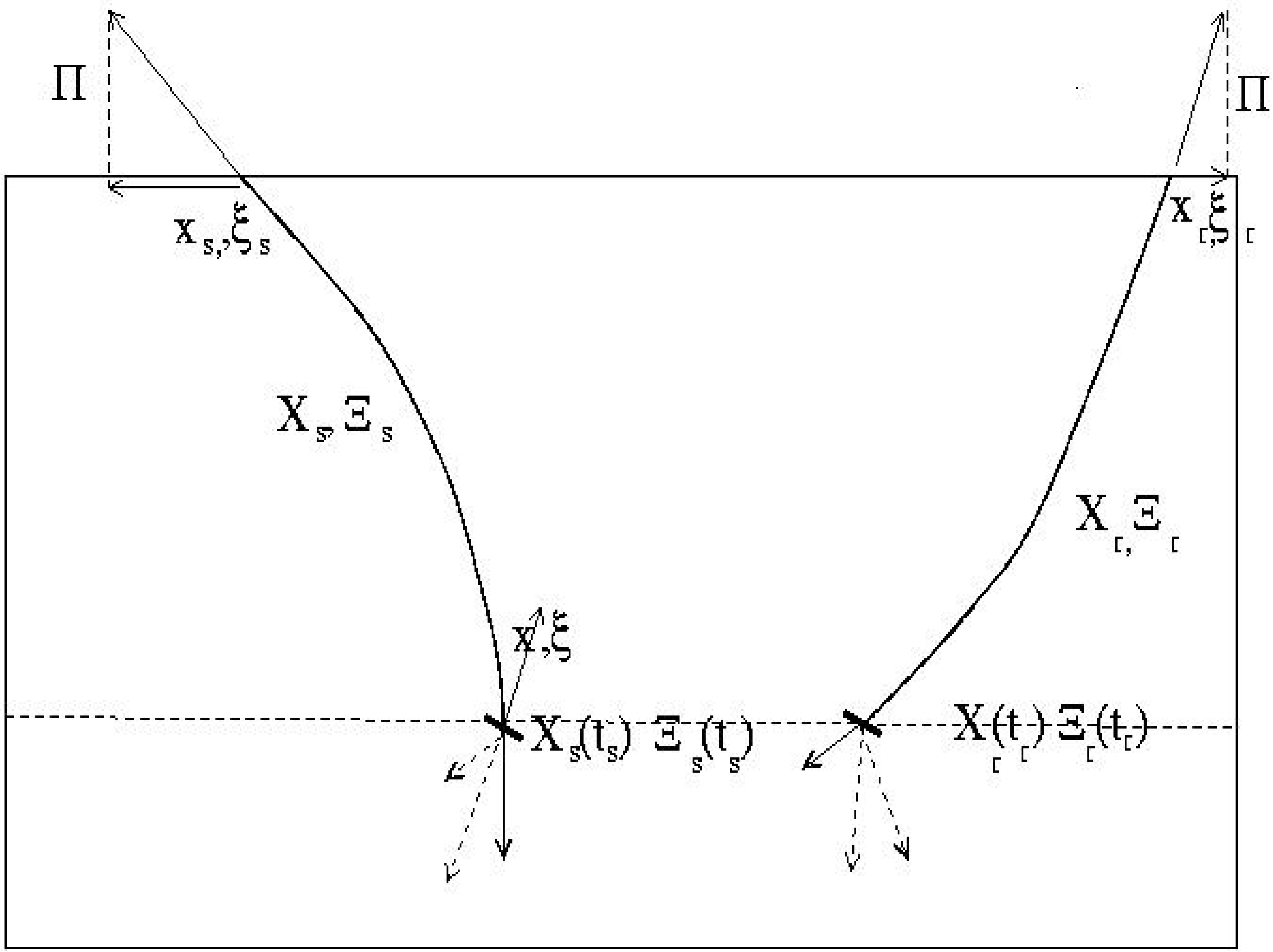
If $\bar{r} \in \mathcal{Z}$, then $(\mathbf{x}, \mathbf{h}, \xi, \nu) \in WF(\bar{r}) \Rightarrow h_3 = 0$. So source and receiver rays in $C_{\bar{F}}$ must terminate at same depth, to hit such a point.

Because of DSR assumption, this fixes the traveltimes t_s, t_r.

Restricted to $\mathcal{Z}$, $C_{\bar{F}}$ is injective.

$\Rightarrow C_{\bar{F}^*\bar{F}} = I$

$\Rightarrow \bar{F}^*\bar{F}$ is ΨDO when restricted to $\mathcal{Z}$.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Quantifying the semblance principle: devise operator W for which

$$\ker W \simeq \mathcal{R}\chi,$$

then minimize a suitable norm of

$$W\bar{G}d^{\mathrm{obs}}.$$

Converts inverse problem to optimization problem. With proper choice of W, ΨDO property $\Rightarrow$ objective is *smooth* $\Rightarrow$ can use Newton and relatives.

Upshot: Claerbout's depth oriented extension appears to offer basis for efficient new algorithms to solve velocity analysis problem - research currently under way in several groups.

Summary:

- quite a bit is known about the imaging problem under "standard hypotheses": mathematics of multipathing imaging (asymptotic inversion, invertible extensions) clarified over last 10 years.
- many imaging situations (eg. near salt) violate "standard hypotheses" grossly - need much better theory
- extension of imaging *via* multiple suppression - some progress, many open questions re non-surface multiples
- velocity analysis - some progress, but still in primitive state mathematically
- almost no progress on underlying nonlinear inverse problem