Extensions and Nonlinear Inverse Scattering

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Outline

1. The acoustic model of reflection seismology
2. Linearization
3. Why Least Squares doesn't work
4. Extensions
5. Annihilators
6. Etc.
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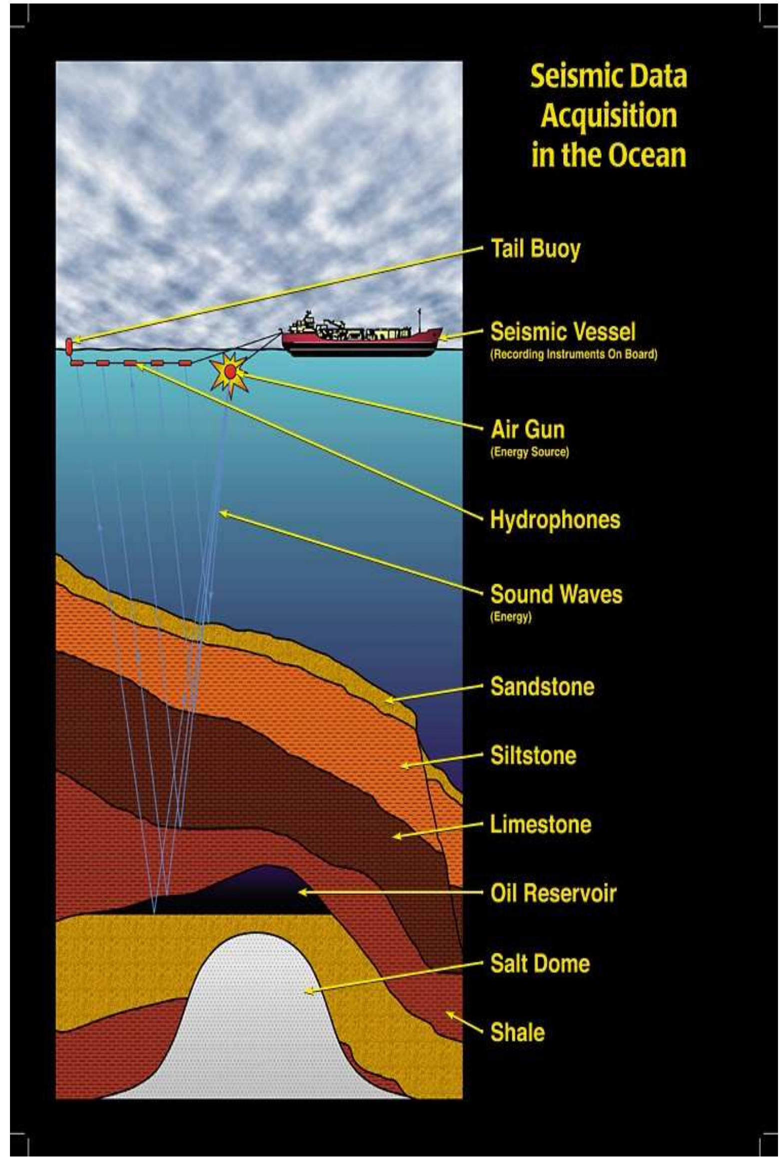
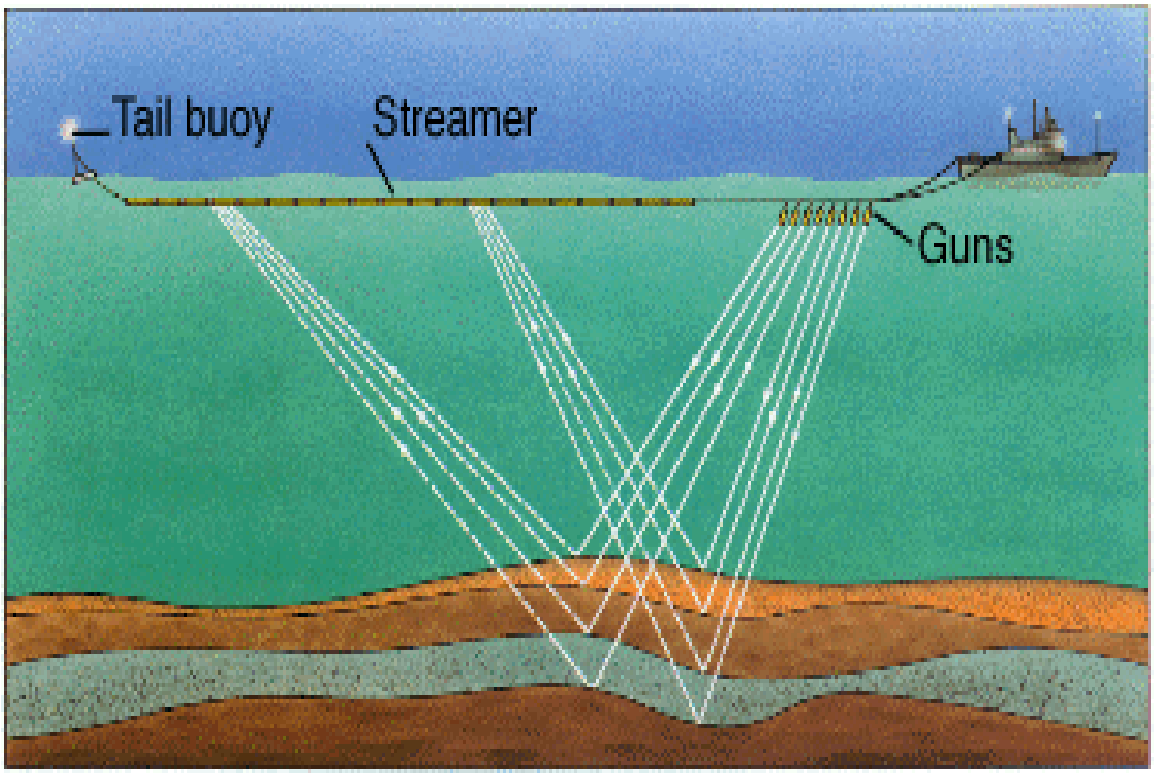
1. The Acoustic Model of Reflection Seismology

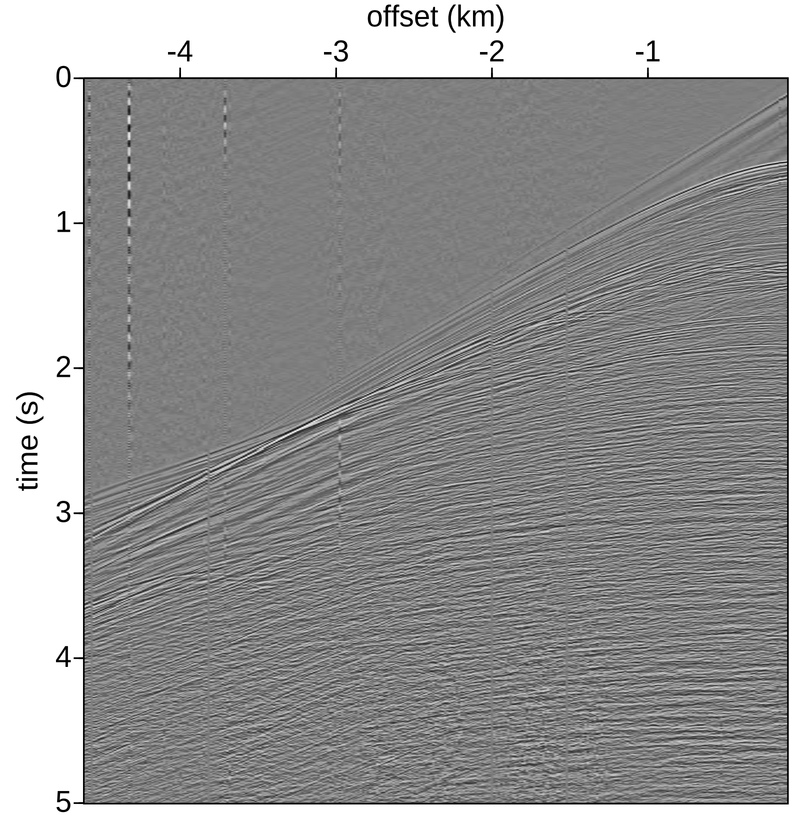
Marine Acquisition
90%+ of all data
collected worldwide





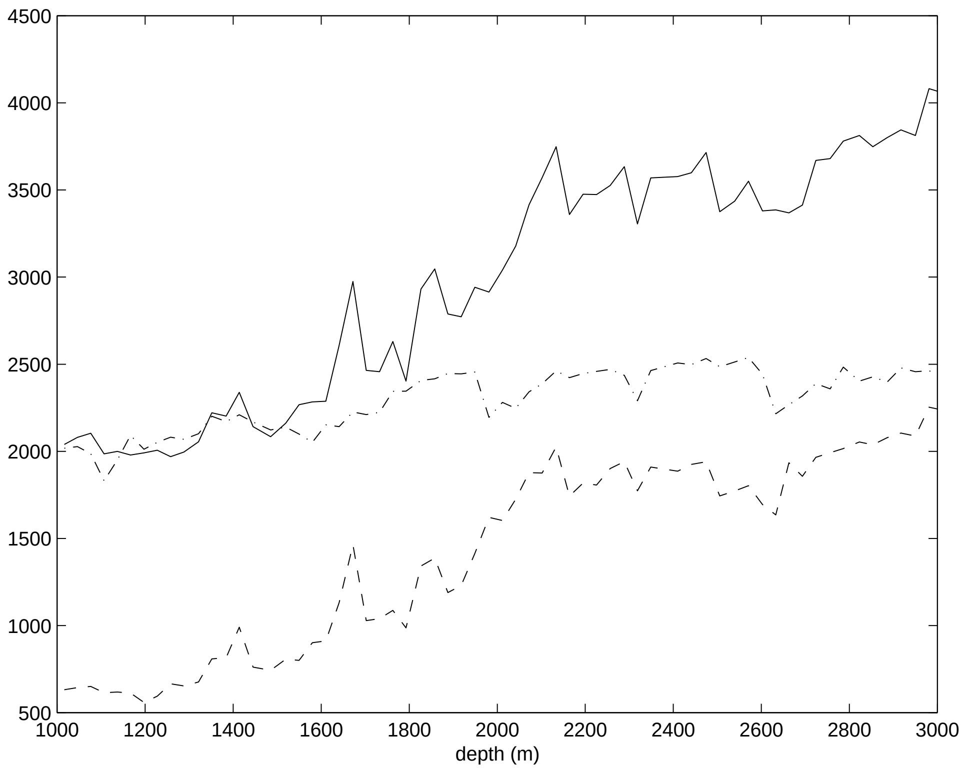
Data parameters: time t, source location $\mathbf{x}_s$, and receiver location $\mathbf{x}_r$, (vector) *half offset* $\mathbf{h} = \frac{\mathbf{x}_r - \mathbf{x}_s}{2}$, scalar half offset $h = |\mathbf{h}|$. Experiment = *shot*, single experiment data = *shot record*.

Typical Marine Record



Shot record, Gulf of Mexico (thanks: Exxon)

Mechanical Characteristics of Sedimentary Rock



Well logs from North Sea borehole. Top curve: compressional wave velocity (m/s); middle curve: density (kg/m^3); bottom curve: shear wave velocity (m/s). (thanks: Mobil R&D, Viking Graben)

Constant Density Acoustic Model

acoustic potential $u(\mathbf{x},t)$ related to pressure p and particle velocity $\mathbf{v}$ by

$$p = \frac{\partial u}{\partial t},\ \mathbf{v} = \frac{1}{\rho}\nabla u$$

Second order wave equation for potential

$$\left(\frac{1}{c(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\mathbf{x},t) = w(t)\delta(\mathbf{x}-\mathbf{x}_s)$$

plus initial, boundary conditions. RHS models localized energy source, "no low frequencies" - *many wavelengths* between source and target. *Useful idealization:* $w(t) = \delta(t)$.

Forward map: $\mathcal{F}[c] \equiv p|_Y$, where $Y = \{(t,\mathbf{x}_r,\mathbf{x}_s) : 0 \le t \le T, ...\}$ is *acquisition manifold*.

2. Linearization

Nonlinear inverse scattering

Inverse problem: given $d \in L^2(Y)$ find $c \in C$ s. t. $\mathcal{F}[c] \simeq d$.

Many difficulties:

- What is C?
- What is $\simeq$?
- If $\simeq$ means "close in L^2", could pose as *least squares* problem: find $c \in C$ to minimize $\|\mathcal{F}[c] - d\|^2$.
- Results of numerical experimentation mixed.
- Theoretical foundation inadequate - few results re relevant properties of $\mathcal{F}$.

(Partly) linearized inverse scattering

Formally, $\mathcal{F}[v(1+r)] \simeq \mathcal{F}[v] + F[v]r$ where $F[\cdot]$ is *linearized forward map* defined by

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta u(\mathbf{x},t) = 2\frac{r(\mathbf{x})}{v^2(\mathbf{x})}\frac{\partial^2 u}{\partial t^2}(\mathbf{x},t)$$

$$F[v]r = \delta p|_Y$$

- basis of most practical data processing procedures.
- Beylkin (1985) and many others: good understanding of the *linear* map $r \mapsto F[v]r$ and the associated *linear* inverse problem for r given v;
- v is no more known than r, inverse problem for $[v,r]$ still nonlinear!

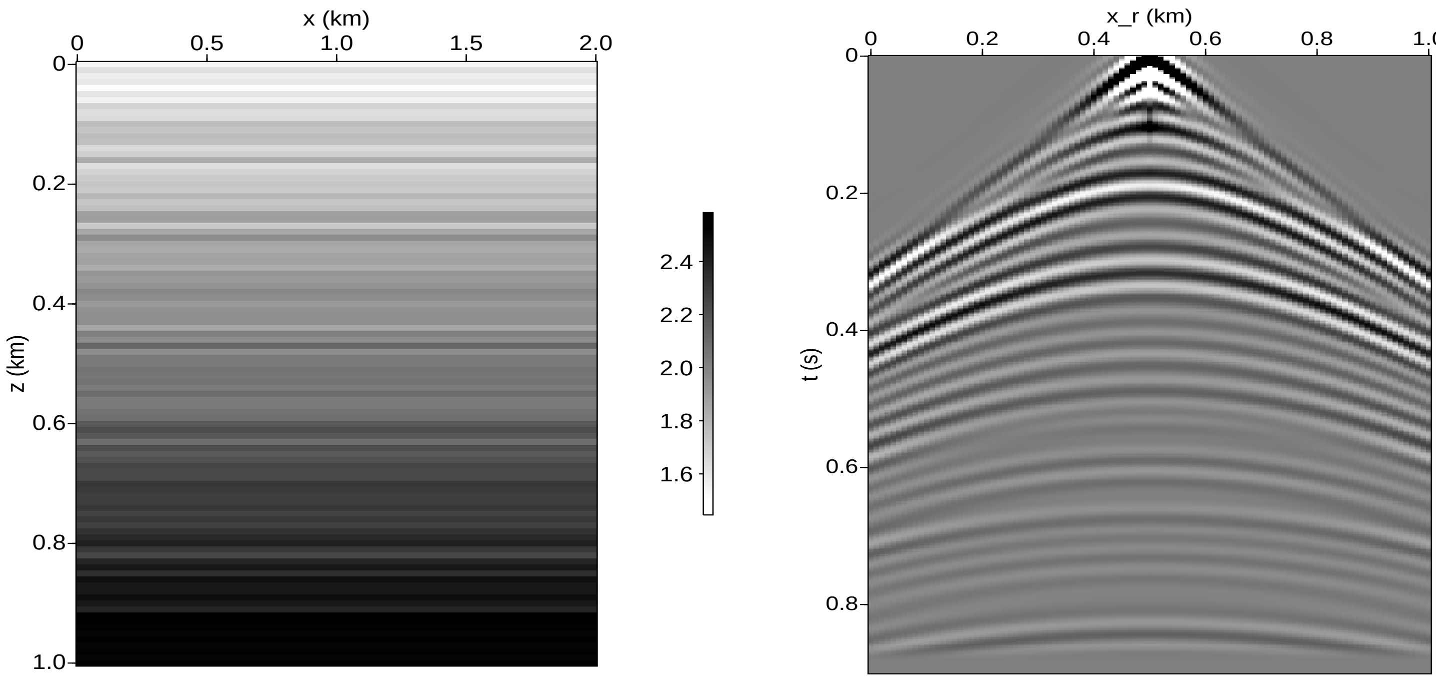
Linearization error

Critical question: If there is any justice $F[v]r$ = directional derivative $D\mathcal{F}[v][vr]$ of $\mathcal{F}$ - but in what sense? Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

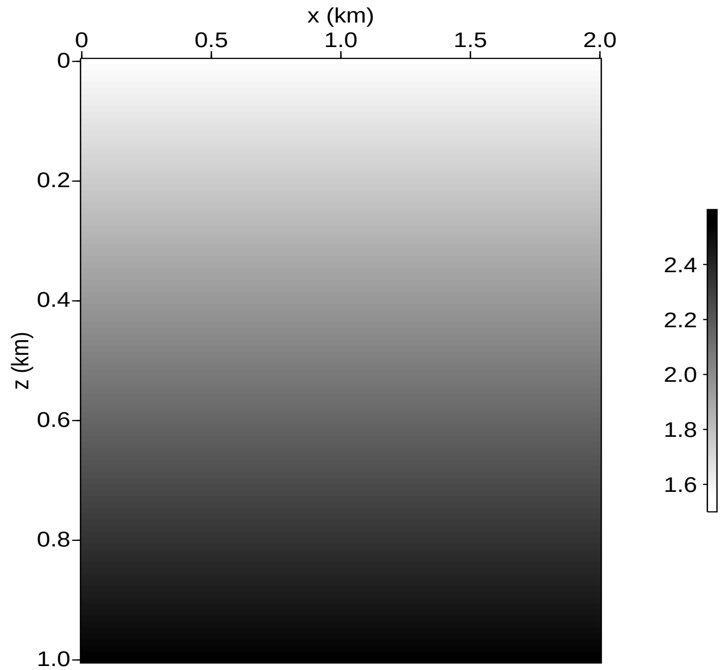
$$\mathcal{F}[v(1+r)] - (\mathcal{F}[v] + F[v]r)$$

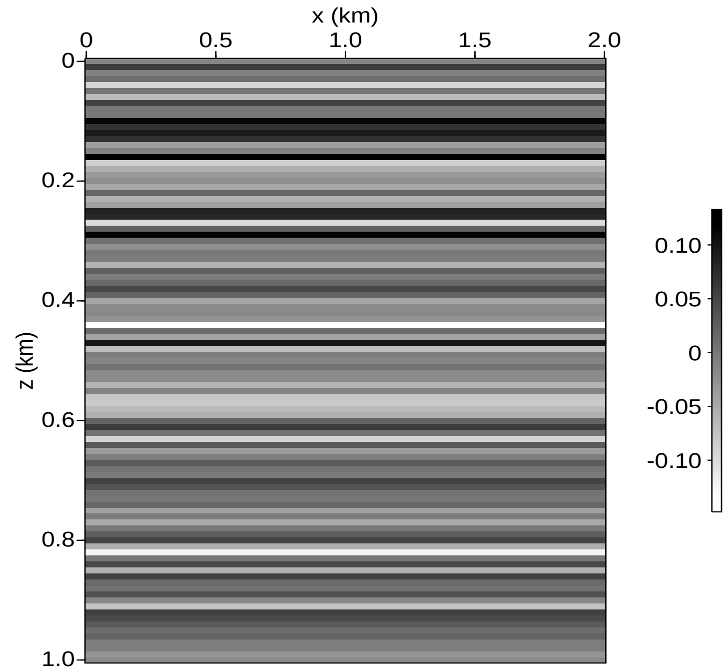
- *small* when v smooth, r rough or oscillatory on wavelength scale - well-separated scales
- *large* when v not smooth and/or r not oscillatory - poorly separated scales

2D finite difference simulation: shot gathers with typical marine seismic geometry. Smooth (linear) $v(x,z)$, oscillatory (random) $r(x,z)$ depending only on z("layered medium"). Source wavelet $w(t)$ = bandpass filter.



Left: Total velocity $c = v(1 + r)$ with smooth (linear) background $v(x, z)$, oscillatory (random) $r(x, z)$. Std dev of r = 5%.
Right: Simulated seismic response ($\mathcal{F}[v(1 + r)]$), wavelet = bandpass filter 4-10-30-45 Hz. Simulator is (2,4) finite difference scheme.





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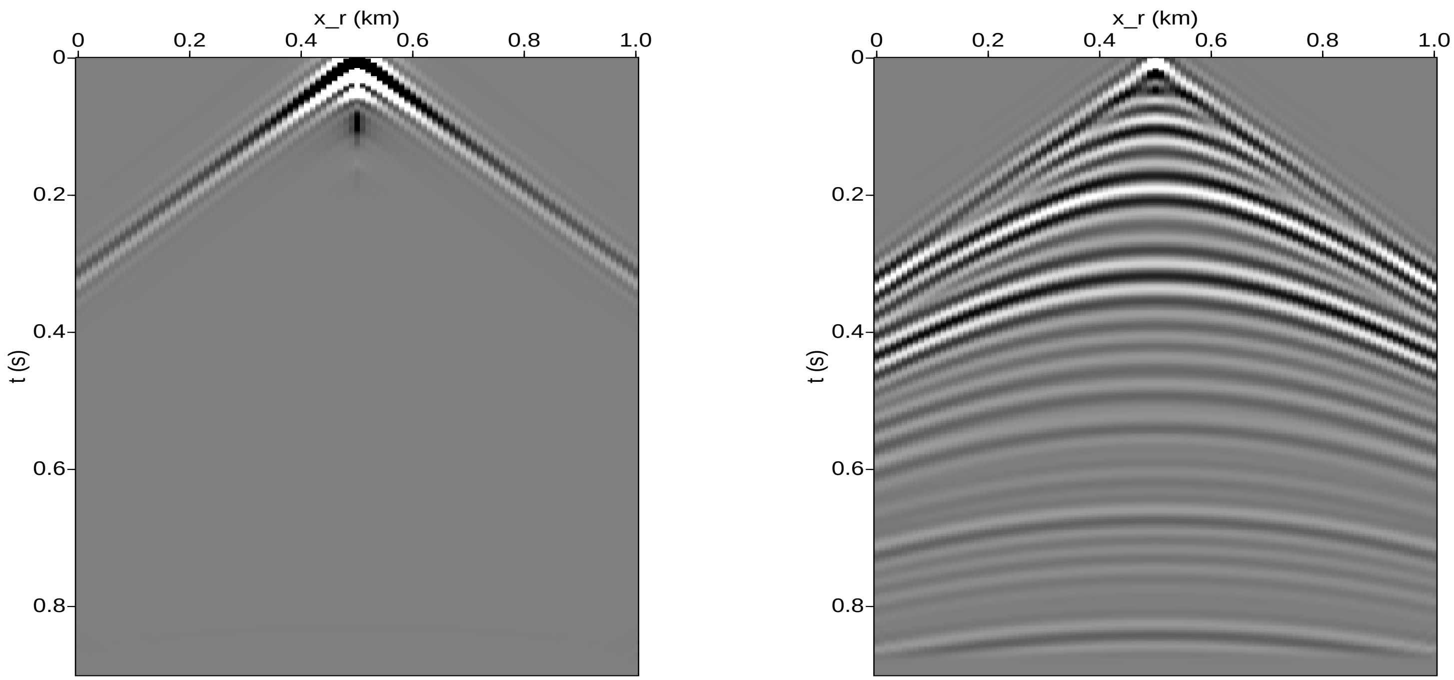
Model in previous slide as smooth background (left, $v(x, z)$) plus rough perturbation (right, $r(x, z)$).



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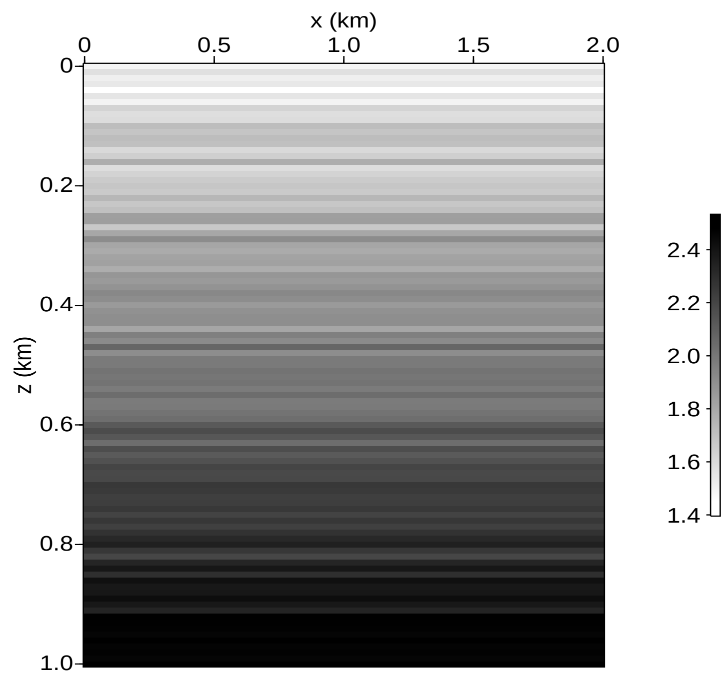
Left: Simulated seismic response of smooth model ($\mathcal{F}[v]$),
Right: Simulated linearized response, rough perturbation of smooth model ($F[v]r$)



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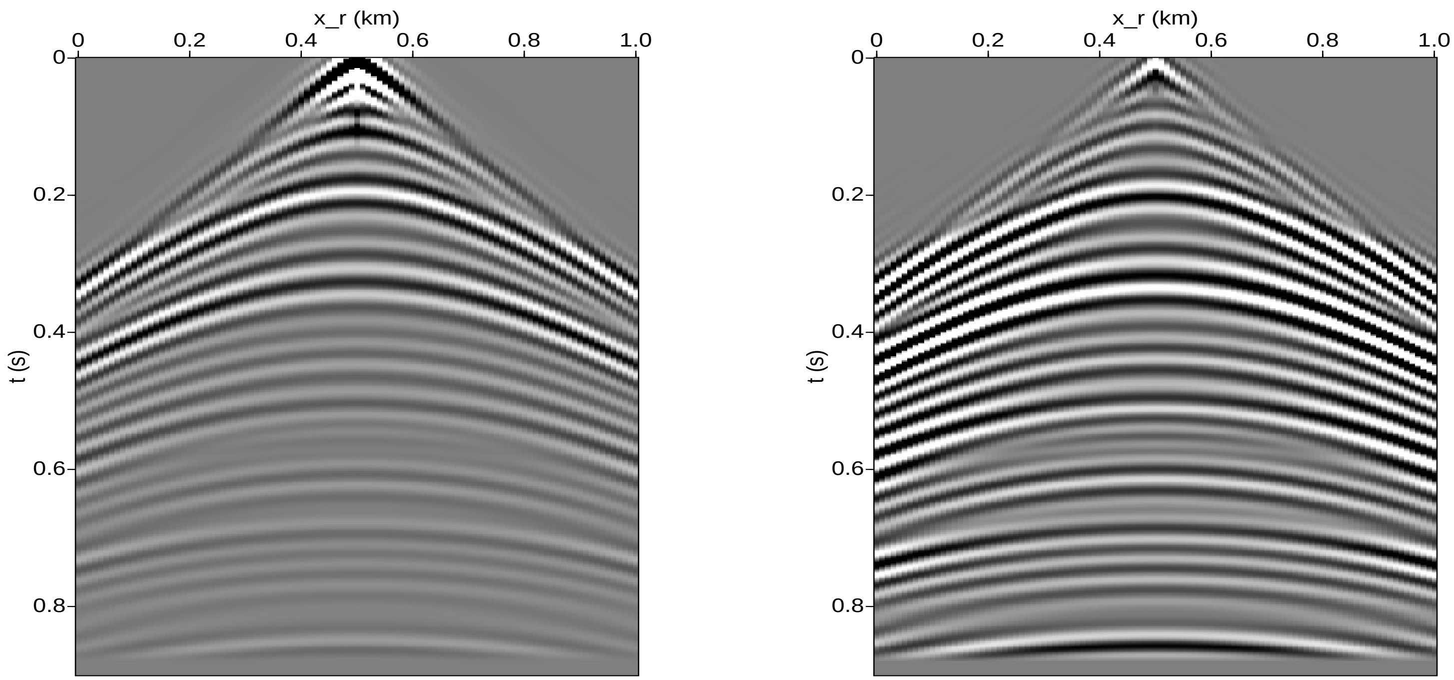
Model in previous slide as rough background (left, $v(x, z)$) plus smooth 5% perturbation ($r(x, z)$).



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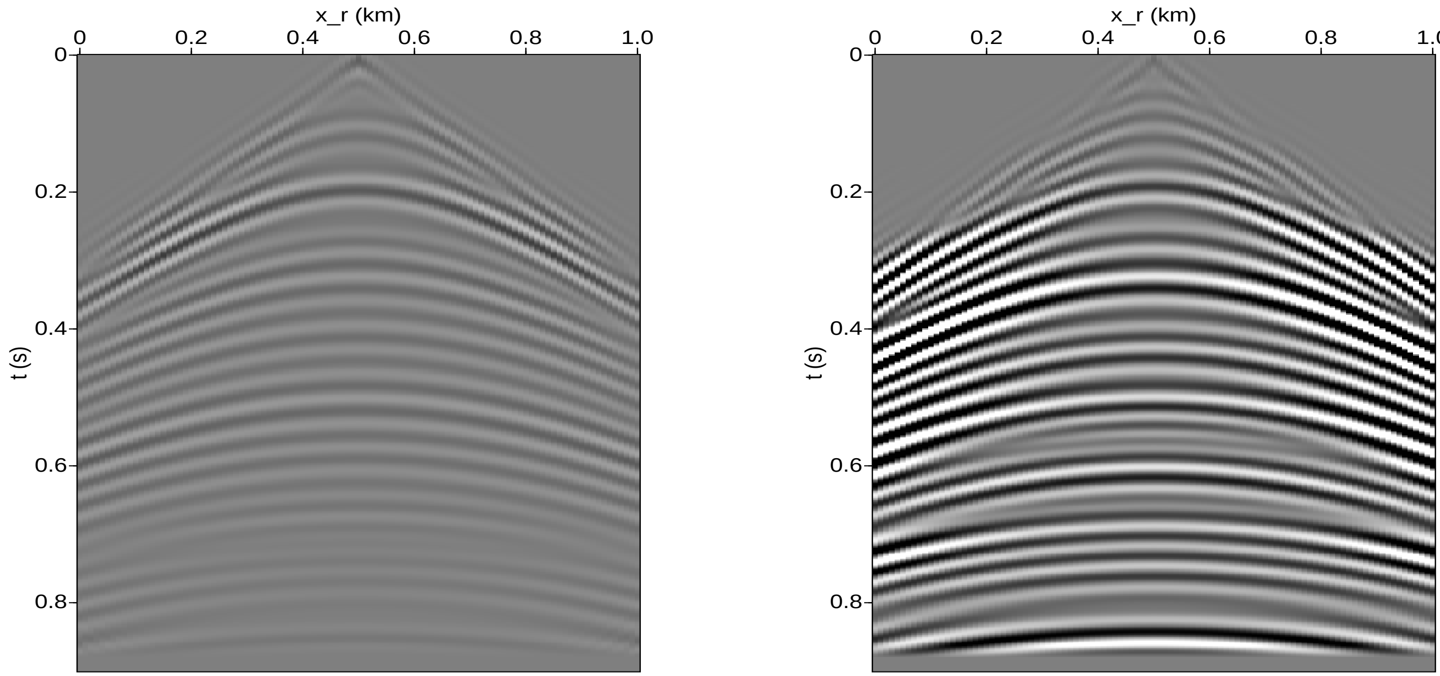
Left: Simulated seismic response of rough model ($\mathcal{F}[v]$),
Right: Simulated linearized response, smooth perturbation of rough model ($F[v]r$)



Left: linearization error ($\mathcal{F}[v(1+r)] - \mathcal{F}[v] - F[v]r$), rough perturbation of smooth background

Right: linearization error, smooth perturbation of rough background (plotted with same grey scale).

Implications

- Some geologies have well-separated scales - cf. sonic logs - linearization-based methods work well there. Other geologies do not - expect trouble!
- v smooth, r oscillatory $\Rightarrow$ $F[v]r$ approximates **primary reflection** = result of wave interacting with material heterogeneity only once (single scattering); error consists of **multiple reflections**, which are "not too large" if r is "not too big", and sometimes can be suppressed.
- v nonsmooth, r smooth $\Rightarrow$ error consists of *time shifts* in waves which are very large perturbations as waves are oscillatory.

No mathematical results are known which justify/explain these observations in any rigorous way, except in 1D.

3. Why Least Squares doesn't work

$$\min_c \|\mathcal{F}[c] - d\|^2$$

- Problems are so large that iterative methods (variants of Newton) are only option (3D: millions of unknowns, billions of equations) $\Rightarrow$ can only find stationary points;
- For any choice of norm in domain, $D\mathcal{F}$ has very poor condition - very large, very small singular values (cf. examples);
- Poor approximation of $\mathcal{F}$ by linearization $\Rightarrow$ poor approximation of least squares function by quadratic;
- Observed behaviour is *nonconvex* $\Rightarrow$ many stationary points exist with large residuals.
- Same remarks apply (and are a bit easier to justify) for *partially linearized least squares* $\min_{v,r} \|F[v]r - (d - \mathcal{F}[v])\|^2$.

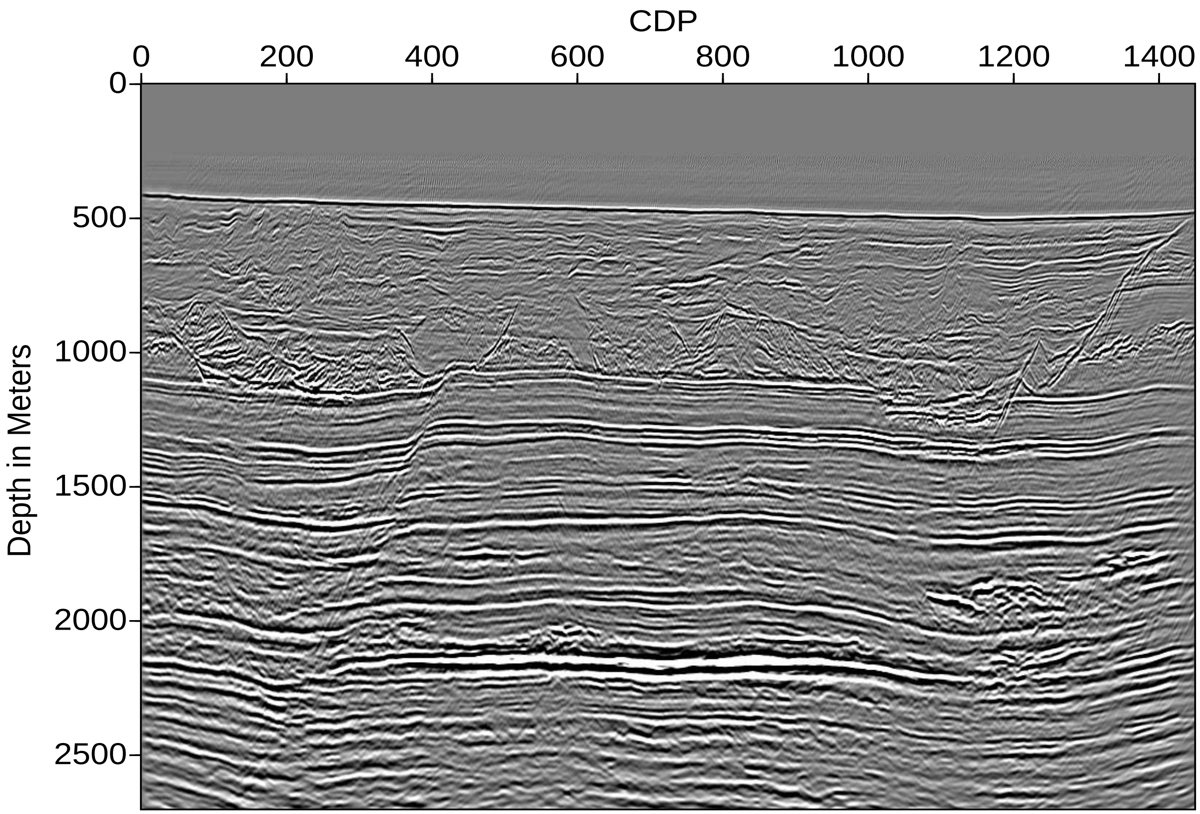
The good news...

We actually know something about $F[v]$, besides its representation when $w(t) = \delta(t)$:

$$F[v]r(t, \mathbf{x}_r, \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int \, dx \int \, d\tau \, G(\mathbf{x}, \mathbf{x}_r, t - \tau) G(\mathbf{x}, \mathbf{x}_s, \tau) \frac{2r(\mathbf{x})}{v^2(\mathbf{x})}$$

- $F[v] : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y)$ (X = Earth) is a *Fourier Integral Operator* associated to a *canonical relation* (Lagrangian submanifold of $T^*(X \times Y)$) (Rakesh, 1988);
- when canonical relation is graph, representation as *Generalized Radon Transform* (Beylkin, 1985) $\Rightarrow$ many practical computations;
- when canonical relation is a graph (Beylkin 1985, Rakesh 1988) and sometimes even when it isn't (Smit, Verdel, tenKroode 1998, Nolan 1997, Stolk 2000), $F[v]^* F[v]$ is *pseudodifferential operator* $\Rightarrow$ construction of *left parametrix* or approximate microlocal inverse.

$$\min_r \|F[v]r - (d - \mathcal{F}[v])\|^2, \text{ given } v$$



Approximate linear least squares solution après Beylkin ("GRT inversion"), Mississippi Canyon, Gulf of Mexico, 2D survey (750 MB, 500 shots). Thanks: Exxon.

4. Extensions

Extended models

Extension of $F[v]$ (aka *extended model*): manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

$$\begin{tikzcd} \mathcal{E}'(\bar{X}) \arrow[r, "{\bar{F}[v]}"] & \mathcal{D}'(Y) \\ \mathcal{E}'(X) \arrow[u, "\chi"] \arrow[r, "{F[v]}"'] & \mathcal{D}'(Y) \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

commutes, i.e.

$$\bar{F}[v]\chi r = F[v]r$$

Extension is "invertible" iff $\bar{F}[v]$ has a *right parametrix* $\bar{G}[v]$, i.e. $I - \bar{F}[v]\bar{G}[v]$is smoothing, or more generally if $\bar{F}[v]\bar{G}[v]$ is pseudodifferential ("inverse except for wrong amplitudes"). Also require existence of a left inverse η for χ: $\eta\chi = \mathrm{id}$.

NB: The trivial extension - $\bar{X} = X, \bar{F} = F$ - is virtually never invertible.

Grand Example

The Standard Extended Model:

- $\bar{X} = X \times H$, $H =$ offset range.
- $\chi r(\mathbf{x},\mathbf{h}) = r(\mathbf{x})$ (so $\bar{r} \in$ range of $\chi \Leftrightarrow$ plots of $\bar{r}(\cdot,\cdot,z,\mathbf{h})$ ("image gathers") appear *flat*)
- $$\bar{F}[v]\bar{r}(\mathbf{x}_r,\mathbf{x}_s,t) = \frac{\partial^2}{\partial t^2}\int\, dx \int\, d\tau\, G(\mathbf{x},\mathbf{x}_r,t-\tau)G(\mathbf{x},\mathbf{x}_s,\tau)\frac{2\bar{r}(\mathbf{x},\mathbf{h})}{v^2(\mathbf{x})}$$

  (recall $\mathbf{h} = (\mathbf{x}_r - \mathbf{x}_s)/2$)

NB: $\bar{F}$ is "block diagonal" - family of operators (FIOs) parametrized by $\mathbf{h}$.

Reformulation of inverse problem

Given d, find v so that $\bar{G}[v]d \in$ the range of χ.

Claim: if v is so chosen, then $[v, r]$ solves partially linearized inverse problem with $r = \eta\bar{G}[v]d$.

Proof: Hypothesis means

$$\bar{G}[v]d = \chi r$$

for some r (whence necessarily $r = \eta\bar{G}[v]d$), so

$$d \simeq \bar{F}[v]\bar{G}[v]d = \bar{F}[v]\chi r = F[v]r$$

Q. E. D.

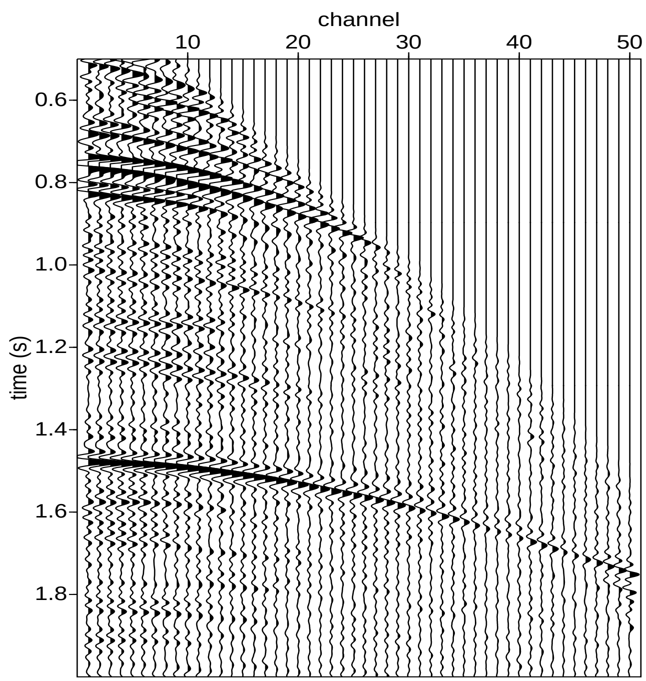
Application: Migration Velocity Analysis

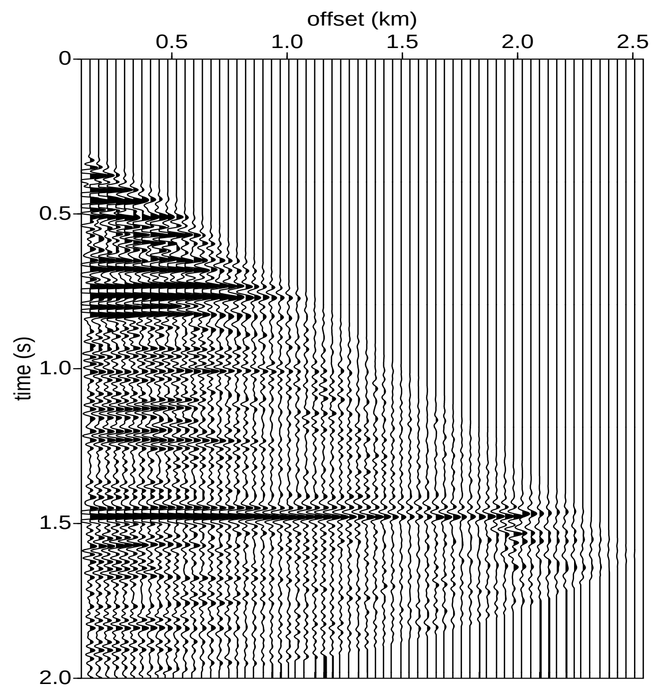
Membership in range of χ is *visually evident*

$\Rightarrow$ industrial practice: adjust parameters of v *by hand* (!) until visual characteristics of $\mathcal{R}(\chi)$ satisfied - "flatten the image gathers".

For the Standard Extended Model, this means: until $\bar{G}[v]d$ is independent of $\mathbf{h}$.

Practically: insist only that $\bar{F}[v]\bar{G}[v]$ be pseudodifferential, so adjust v until $\bar{G}[v]d$ is "smooth" in $\mathbf{h}$.





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Left: shot record (d) from North Sea survey (thanks: Shell Research), lightly pre-processed.
Right: restriction of $\bar{G}[v]d^{\text{obs}}$ to $x, y =$ const (function of depth, offset): shows rel. sm'ness in h (offset) for properly chosen v.

5. Annihilators

Automating the reformulation

Suppose $W : \mathcal{E}'(\bar{X}) \rightarrow \mathcal{D}'(Z)$ annihilates range of χ:

$$\mathcal{E}'(X) \stackrel{\chi}{\rightarrow} \mathcal{E}'(\bar{X}) \stackrel{W}{\rightarrow} \mathcal{D}'(Z) \rightarrow 0$$

and moreover W is bounded on $L^2(\bar{X})$. Then

$$J[v;d] = \frac{1}{2}\|W\bar{G}[v]d\|^2$$

minimized when $[v, \eta\bar{G}[v]d]$ solves partially linearized inverse problem.

Construction of *annihilator* of $\mathcal{R}(F[v])$ (Guillemin, 1985 - cf deHoop's talks):

$$d \in \mathcal{R}(F[v]) \Leftrightarrow \bar{G}[v]d \in \mathcal{R}(\chi) \Leftrightarrow W\bar{G}[v]d = 0$$

Annihilators, annihilators everywhere...

For Standard Extended Model, several popular choices:

- $$W = (I - \Delta)^{-\frac{1}{2}} \nabla_{\mathbf{h}}$$

  ("differential semblance" - WWS, 1986)

- $$W = I - \frac{1}{|H|} \int dh$$

  ("stack power" - Toldi, 1985)

- $$W = I - \chi F[v]^{\dagger} \bar{F}[v]$$

  $\Rightarrow$ minimizing $J[v, d]$ equivalent to least squares.

But not many are good for much...

Since *problem is huge*, only W giving rise to differentiable $v \mapsto J[v,d]$ are useful - must be able to use Newton!!! Once again, idealize $w(t) = \delta(t)$.

Theorem (Stolk & WWS, 2003): $v \mapsto J[v,d]$ smooth $\Leftrightarrow$ W pseudodifferential.

i.e. only *differential semblance* gives rise to smooth optimization problem, regardless of source bandwidth.

NB: Least squares embedded in larger family of optimization formulations, some (others) of which are tractable.

Numerical evidence using synthetic and field data: WWS et al., Chauris & Noble 2001, Mulder & tenKroode 2002. deHoop et al. 2004.

6. Etc.

Invertible Extensions

- Beylkin (1985), Rakesh (1988): if $\|\nabla^2 v\|_{C^0}$ “not too big” (no caustics appear), then the Standard Extended Model is invertible.
- Nolan & WWS 1997, Stolk & WWS 2004: if $\|\nabla^2 v\|_{C^0}$ is too big (caustics, multipathing), Standard Extended Model is *not* invertible! Not in any version - common offset, common source, common scattering angle,...
- Stolk & deHoop 2001: *Claerbout extension* is invertible under much weaker condition (absence of turning rays).
- WWS, Stolk, Biondi 2003: generalized Claerbout extension to accommodate turning rays.

Beyond Born

Nonlinear effects not included in linearized model: *multiple reflections*. Conventional approach: treat as *coherent noise*, attempt to eliminate - active area of research going back 40+ years, with recent important developments.

Why not model this "noise"?

Proposal: *nonlinear extensions* with $F[v]r$ replaced by $\mathcal{F}[c]$. Create annihilators in same way (now also nonlinear), optimize differential semblance.

Nonlinear analog of Standard Extended Model appears to be *invertible* - in fact extended nonlinear inverse problem is *underdetermined*.

Open problems: no theory. Also must determine $w(t)$ (Delprat & Lailly 2003).

And so on...

- Elasticity: theory of Born inversion at smooth background in good shape (Beylkin & Burridge 1988, deHoop & Bleistein 1997). Theory of extensions, annihilators, differential semblance partially complete (Brandsberg-Dahl et al 2003).
- Anisotropy - see deHoop's talk, this meeting.
- Anelasticity - in the sedimentary section, $Q = 100 - 1000$, lower in gassy sediments and near surface. No results.
- Source determination - actually always an issue. Some success in casting as an inverse problem - Minkoff & WWS 1997, Routh et al SEG 2003.
- ...

Conclusion

- Least error formulation of (waveform) reflection seismic inverse problem *intractable* - very irregular with large residual stationary points $\Rightarrow$ *no influence on practice*.
- Linearized *extended models* provide framework for both (industry standard) interpretive velocity analysis and automated techniques based on construction of *range annihilators*.
- Only *(pseudo)differential annihilators* yield smooth objective functions.
- Not all extensions suitable for use in "complex structure" (strong refraction).
- May be able to account for more nonlinearity (multiple reflections) via nonlinear extensions.

Thanks to...

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The Organizers, for inviting me.

`http://www.trip.caam.rice.edu`

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