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# Mathematics of Seismic Imaging

## Part III

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# Wave Equation Migration

Techniques for computing  $F[v]^*$ :

(i) Reverse time

(ii) Reverse depth

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# Reverse Time Migration, Zero Offset

Start with the zero-offset case - easier, but only if you replace it with the exploding reflector model, which replaces  $F[v]$  by

$$\tilde{F}[v]r(\mathbf{x}_s, t) = w(\mathbf{x}_s, t), \quad \mathbf{x}_s \in X_s, 0 \leq t \leq T$$

$$\left( \frac{4}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) w = \delta(t) \frac{2r}{v^2}, \quad w \equiv 0, t < 0$$

To compute the adjoint, start with its definition: choose  $d \in \mathcal{E}(X_s \times (0, T))$ , so that

$$\begin{aligned} & \langle \tilde{F}[v]^* d, r \rangle = \langle d, \tilde{F}[v]r \rangle \\ &= \int_{X_s} dx_s \int_0^T dt d(\mathbf{x}_s, t) w(\mathbf{x}_s, t) \end{aligned}$$

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The only thing you know about  $w$  is that it solves a wave equation with  $r$  on the RHS. To get this fact into play, (i) rewrite the integral as a space-time integral:

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \int_{X_s} dx_s d(\mathbf{x}_s, t) \delta(\mathbf{x} - \mathbf{x}_s) w(\mathbf{x}, t)$$

(ii) write the other factor in the integrand as the image of a field  $q$  under the (adjoint of the) wave operator (it's self-adjoint), that is,

$$\left( \frac{4}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) q(\mathbf{x}, t) = \int_{X_s} dx_s d(\mathbf{x}_s, t) \delta(\mathbf{x} - \mathbf{x}_s)$$

so

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \left[ \left( \frac{4}{v^2(\mathbf{x})} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) q(\mathbf{x}, t) \right] w(\mathbf{x}, t)$$

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(iii) integrate by parts

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \left[ \left( \frac{4}{v^2(\mathbf{x})} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) w(\mathbf{x}, t) \right] q(\mathbf{x}, t)$$

which works if  $q \equiv 0$ ,  $t > T$  (*final value condition*); (iv) use the wave equation for  $w$

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \frac{2}{v(\mathbf{x})^2} r(\mathbf{x}) \delta(t) q(\mathbf{x}, t)$$

(v) observe that you have computed the adjoint:

$$= \int_{\mathbf{R}^3} dx r(\mathbf{x}) \left[ \frac{2}{v(\mathbf{x})^2} q(\mathbf{x}, 0) \right] = \langle r, \tilde{F}[v]^* d \rangle$$

i.e.

$$\tilde{F}[v]^* d = \frac{2}{v(\mathbf{x})^2} q(\mathbf{x}, 0)$$

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Summary of the computation, with the usual description:

- Use that data as sources, backpropagate in time - i.e. solve the final value (“reverse time”) problem

$$\left( \frac{4}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) q(\mathbf{x}, t) = \int_{X_s} dx_s d(\mathbf{x}_s, t) \delta(\mathbf{x} - \mathbf{x}_s), \quad q \equiv 0, \quad t > T$$

- read out the “image” (= adjoint output) at  $t = 0$ :

$$\tilde{F}[v]^* d = \frac{2}{v(\mathbf{x})^2} q(\mathbf{x}, 0)$$

**Note:** The adjoint (time-reversed) field  $q$  is *not* the physical field ( $\delta u$ ) run backwards in time, contrary to some imputations in the literature.

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## Historical Remarks

- Known as “two way reverse time finite difference poststack migration” in geophysical literature (Whitmore, 1982)
- uses full (two way) wave equation, propagates adjoint field backwards in time, generally implemented using finite difference discretization.
- Same as “adjoint state method”, Lions 1968, Chavent 1974 for control and inverse problems for PDEs - much earlier for control of ODEs - Lailly, Tarantola '80s.
- My buddy Tapia says: all you're doing is transposing a matrix! True (after discretization), but it's important that these matrices are triangular, so can be implemented by recursions - forward for simulation, backwards for adjoint.

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## Reverse Time Migration, Prestack

A slightly messier computation computes the adjoint of  $F[v]$  (i.e. multioffset or *prestack* migration):

$$F[v]^* d(\mathbf{x}) = -\frac{2}{v(\mathbf{x})} \int dx_s \int_0^T dt \left( \frac{\partial q}{\partial t} \nabla^2 u \right) (\mathbf{x}, t; \mathbf{x}_s)$$

where *adjoint field*  $q$  satisfies  $q \equiv 0, t \geq T$  and

$$\left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) q(\mathbf{x}, t; \mathbf{x}_s) = \int dx_r d(\mathbf{x}_r, t; \mathbf{x}_s) \delta(\mathbf{x} - \mathbf{x}_r)$$



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## Proof

$$\begin{aligned} & \langle F[v]^* d, r \rangle = \langle d, F[v] r \rangle \\ &= \int \int dx_s dx_r \int_0^T dt d(\mathbf{x}_r, t; \mathbf{x}_s) \frac{\partial \delta u}{\partial t}(\mathbf{x}_r, t; \mathbf{x}_s) \\ &= \int dx_s \int dx \int_0^T dt \left\{ \int dx_r d(\mathbf{x}_r, t; \mathbf{x}_s) \delta(\mathbf{x} - \mathbf{x}_r) \right\} \frac{\partial \delta u}{\partial t}(\mathbf{x}, t; \mathbf{x}_s) \\ &= \int dx_s \int dx \int_0^T dt \left[ \left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) q \right] \frac{\partial \delta u}{\partial t}(\mathbf{x}, t; \mathbf{x}_s) \end{aligned}$$

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$$= - \int dx_s \int dx \int_0^T dt \left[ \left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta u \right] \frac{\partial q}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

(boundary terms in integration by parts vanish because (i)  $\delta u \equiv 0$ ,  $t \ll 0$ ; (ii)  $q \equiv 0$ ,  $t \gg 0$ ; (iii) both vanish for large  $\mathbf{x}$ , at each  $t$ )

$$= - \int dx_s \int dx \int_0^T dt \left( \frac{2r}{v^2} \frac{\partial^2 u}{\partial t^2} \frac{\partial q}{\partial t} \right) (\mathbf{x}, t; \mathbf{x}_s)$$

$$= - \int dx_s \int dx r(\mathbf{x}) \frac{2}{v^2(\mathbf{x})} \int_0^T dt \left( \frac{\partial^2 u}{\partial t^2} \frac{\partial q}{\partial t} \right) (\mathbf{x}, t; \mathbf{x}_s)$$

$$= \langle r, F[v]^* d \rangle$$

**q.e.d.**

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# Implementation

Algorithm: finite difference or finite element discretization in  $\mathbf{x}$ , finite difference time stepping.

- For each  $\mathbf{x}_s$ , solve wave equation for  $u$  forward in  $t$ , record final ( $t=T$ ) Cauchy data, also (for example) Dirichlet boundary data.
- Step  $u$  and  $q$  backwards in time together; at each time step, data serves as source for  $q$  (“backpropagate data”)
- During backwards time stepping, accumulate (approximations to)

$$Q(\mathbf{x})_+ = \frac{2}{v^2(\mathbf{x})} \int_0^T dt \left( \frac{\partial^2 u}{\partial t^2} \frac{\partial q}{\partial t} \right) (\mathbf{x}, t; \mathbf{x}_s)$$

(“crosscorrelate reference and backpropagated field”).

- next  $\mathbf{x}_s$  - after last  $\mathbf{x}_s$ ,  $F[v]^* d = Q$ .

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# Reverse Depth Migration, Zero Offset

aka: depth extrapolation, downward continuation, or simply “wave equation migration”.

Introduced by Claerbout, early 70’s (“swimming pool equation”). Again, assume exploding reflector model:

$$\tilde{F}[v]r(\mathbf{x}_s, t) = w(\mathbf{x}_s, t), \quad \mathbf{x}_s \in X_s, 0 \leq t \leq T$$

$$\left( \frac{4}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) w = \delta(t) \frac{2r}{v^2}, \quad w \equiv 0, t < 0$$

Basic idea: 2nd order wave equation permits waves to move in all directions, but waves carrying reflected energy are (mostly) moving *up*. Should satisfy a 1st order equation for wave motion in one direction.

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## Coming up...

For the moment use 2D notation  $\mathbf{x} = (x, z)$  etc. Write wave equation as evolution equation in  $z$ :

$$\frac{\partial^2 w}{\partial z^2} - \left( \frac{4}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) w = -\delta(t) \frac{2r}{v^2}$$

Suppose that you could take the square root of the operator in parentheses - call it  $B$ . Then the LHS of the wave equation becomes

$$\left( \frac{\partial}{\partial z} - B \right) \left( \frac{\partial}{\partial z} + B \right) w = -\delta(t) \frac{2r}{v^2}$$

so setting  $\tilde{w} = \left( \frac{\partial}{\partial z} + B \right) w$  you get

$$\left( \frac{\partial}{\partial z} - B \right) \tilde{w} = -\delta(t) \frac{2r}{v^2}$$

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## Some issues

This *might* be the required equation for upcoming waves.

Two major problems: (i) how the  $\hbar^{-1}$  do you take the square root of a PDO?

(ii) what guarantees that the equation just written governs upcoming waves?

Answers to be found in the theory of  $\Psi$ DOs!

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## Classical $\Psi$ DOs

Important *subclass* of *classical*  $\Psi$ DOs: those whose (“classical”) symbols have asymptotic expansions:

$$p(\mathbf{x}, \boldsymbol{\xi}) \sim \sum_{j \leq m} p_j(\mathbf{x}, \boldsymbol{\xi}), \quad |\boldsymbol{\xi}| \rightarrow \infty$$

in which  $p_j$  is *homogeneous in  $\boldsymbol{\xi}$  of degree  $j$* :

$$p_j(\mathbf{x}, \tau \boldsymbol{\xi}) = \tau^j p_j(\mathbf{x}, \boldsymbol{\xi}), \quad \tau, |\boldsymbol{\xi}| \geq 1$$

The *principal symbol* is the homogeneous term of highest degree, i.e.  $p_m$  above.

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## Products of $\Psi$ DOs are $\Psi$ DOs.

Classical  $\Psi$ DOs have more complete *calculus*, including prescriptions for “computing” adjoints, products, and the like. From now on unless otherwise stated, all  $\Psi$ DOs are classical.

Product rule for  $\Psi$ DOs: if  $p^1, p^2$  are classical,

$$p^1(\mathbf{x}, \boldsymbol{\xi}) = \sum_{j \leq m^1} p_j^1(\mathbf{x}, \boldsymbol{\xi}), \quad p^2(\mathbf{x}, \boldsymbol{\xi}) = \sum_{j \leq m^2} p_j^2(\mathbf{x}, \boldsymbol{\xi})$$

then so is  $p^1(\mathbf{x}, D)p^2(\mathbf{x}, D)$ , and its principal symbol is  $p_{m^1}^1(\mathbf{x}, \boldsymbol{\xi})p_{m^2}^2(\mathbf{x}, \boldsymbol{\xi})$ , and there is an algorithm for computing the rest of the expansion.

In an open neighborhood  $X \times \Xi$  of  $(\mathbf{x}_0, \boldsymbol{\xi}_0)$ , symbol of  $p^1(\mathbf{x}, D)p^2(\mathbf{x}, D)$  depends only on symbols of  $p^1, p^2$  in  $X \times \Xi$ .

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Consequence: if  $a(\mathbf{x}, D)$  has an asymptotic expansion and is of order  $m \in \mathbf{R}$ , and  $a_m(\mathbf{x}_0, \boldsymbol{\xi}_0) > 0$  in  $\mathcal{P} \subset \mathbf{R}^n \times \mathbf{R}^n - 0$ , then there exists  $b(\mathbf{x}, D)$  of order  $m/2$  with asymptotic expansion for which

$$(a(\mathbf{x}, D) - b(\mathbf{x}, D)b(\mathbf{x}, D))u \in \mathcal{E}(\mathbf{R}^n)$$

for any  $u \in \mathcal{E}'(\mathbf{R}^n)$  with  $WF(u) \subset \mathcal{P}$ .

Moreover,  $b_{m/2}(\mathbf{x}, \boldsymbol{\xi}) = \sqrt{a_m(\mathbf{x}, \boldsymbol{\xi})}$ ,  $(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{P}$ . Will call  $b$  a *microlocal square root* of  $a$ .

Similar construction: if  $a(\mathbf{x}, \boldsymbol{\xi}) \neq 0$  in  $\mathcal{P}$ , then there is  $c(\mathbf{x}, D)$  of order  $-m$  so that

$$c(\mathbf{x}, D)a(\mathbf{x}, D)u - u, \quad a(\mathbf{x}, D)c(\mathbf{x}, D)u - u \in \mathcal{E}(\mathbf{R}^n)$$

for any  $u \in \mathcal{E}'(\mathbf{R}^n)$  with  $WF(u) \subset \mathcal{P}$ .

Moreover,  $c_{-m}(\mathbf{x}, \boldsymbol{\xi}) = 1/a_m(\mathbf{x}, \boldsymbol{\xi})$ ,  $(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{P}$ . Will call  $c$  a *microlocal inverse* of  $a$ .

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## Application: the Square Root Operator

$$a(x, z, D_t, D_x) = \frac{\partial^2}{\partial x^2} - \frac{4}{v(x, z)^2} \frac{\partial^2}{\partial t^2} = \frac{4}{v(x, z)^2} D_t^2 - D_x^2$$

is

$$a(x, z, \tau, \xi) = \frac{4}{v(x, z)^2} \tau^2 - \xi^2$$

For  $\delta > 0$ , set

$$\mathcal{P}_\delta(z) = \left\{ (x, t, \xi, \tau) : \frac{4}{v(x, z)^2} \tau^2 > (1 + \delta) \xi^2 \right\}$$

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## The SSR Operator

Then according to the last slide, there is an order 1  $\Psi$ DO-valued function of  $z$ ,  $b(x, z, D_t, D_x)$ , with principal symbol

$$b_1(x, z, \tau, \xi) = \sqrt{\frac{4}{v(x, z)^2} \tau^2 - \xi^2} = \tau \sqrt{\frac{4}{v(x, z)^2} - \frac{\xi^2}{\tau^2}}, \quad (x, t, \xi, \tau) \in \mathcal{P}_\delta(z)$$

for which  $a(x, z, D_t, D_x)u \simeq b(x, z, D_t, D_x)b(x, z, D_t, D_x)u$  if  $WF(u) \subset \mathcal{P}_\delta(z)$ .

$b$  is the world-famous **single square root** (“SSR”) operator - see Claerbout, IEI.

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## The SSR Assumption

To what extent has this construction factored the wave operator:

$$\begin{aligned} & \left( \frac{\partial}{\partial z} - ib(x, z, D_x, D_t) \right) \left( \frac{\partial}{\partial z} + ib(x, z, D_x, D_t) \right) \\ &= \frac{\partial^2}{\partial z^2} + b(x, z, D_x, D_t)b(x, z, D_x, D_t) + \frac{\partial b}{\partial z}(x, z, D_x, D_t) \end{aligned}$$

**SSR Assumption:** For some  $\delta > 0$ , the wavefield  $w$  satisfies

$$(x, z, t, \xi, \zeta, \tau) \in WF(w) \Rightarrow (x, t, \xi, \tau) \in \mathcal{P}_\delta(z) \text{ and } \zeta\tau > 0$$

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This statement has a ray-theoretic interpretation (which will eventually make sense): rays carrying significant energy are nowhere horizontal. Along any such ray,  $z$  decreases as  $t$  increases - *coming up!*

$$\tilde{w}(x, z, t) = \left( \frac{\partial}{\partial z} + ib(x, z, D_x, D_t) \right) w(x, z, t)$$

$$b(x, z, D_x, D_t)b(x, z, D_x, D_t)w \simeq \left( \frac{4}{v(x, z)^2} D_t^2 - D_x^2 \right) w$$

with a smooth error, so

$$\begin{aligned} \left( \frac{\partial}{\partial z} - ib(x, z, D_x, D_t) \right) \tilde{w}(x, z, t) &= -\frac{2r(x, z)}{v(x, z)^2} \delta(t) \\ &+ i \left( \frac{\partial}{\partial z} b(x, z, D_x, D_t) \right) w(x, z, t) \end{aligned}$$

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(since  $b$  depends on  $z$ , the  $z$  deriv. does not commute with  $b$ ). So  $\tilde{w} = \tilde{w}_0 + \tilde{w}_1$ , where

$$\left( \frac{\partial}{\partial z} - ib(x, z, D_x, D_t) \right) \tilde{w}_0(x, z, t) = -\frac{2r(x, z)}{v(x, z)^2} \delta(t)$$

(this is the **SSR modeling equation**)

$$\left( \frac{\partial}{\partial z} - ib(x, z, D_x, D_t) \right) \tilde{w}_1(x, z, t) = i \left( \frac{\partial}{\partial z} b(x, z, D_x, D_t) \right) w(x, z, t)$$

Claim:  $WF(\tilde{w}_1) \subset WF(w)$ . Granted this  $\Rightarrow WF(\tilde{w}_0) \subset WF(w)$  also.

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Upshot: SSR modeling

$$\tilde{F}_0[v]r(x_s, z_s, t) = \tilde{w}_0(x_s, z_s, t)$$

produces the same singularities (i.e. the same waves) as exploding reflector modeling, so is as good a basis for migration.

SSR migration: assume that sources all lie on  $z_s = 0$ .

$$\begin{aligned} < \tilde{F}_0[v]^* d, r > = < d, \tilde{F}_0[v] r > \\ &= \int dx_s \int dt d(x_s, t) \tilde{w}_0(x_s, 0, t) \end{aligned}$$

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$$= \int dx_s \int dt \int dz d(\bar{x}_s, t) \delta(z) \tilde{w}_0(x_s, z, t)$$

Define the adjoint field  $q$  by

$$\left( \frac{\partial}{\partial z} - b(x, z, D_x, D_t) \right) q(x, z, t) = d(x, t) \delta(z), \quad q(x, z, t) \equiv 0, \quad z < 0$$

which is equivalent to solving the initial value problem

$$\left( \frac{\partial}{\partial z} - ib(x, z, D_x, D_t) \right) q(x, z, t) = 0, \quad z > 0; \quad q(x, 0, t) = d(x, t)$$

Insert in expression for inner product, integrate by parts, use self-adjointness of  $b$ , get

$$\langle d, \tilde{F}_0[v]r \rangle = \int dx \int dz \frac{2r(x, z)}{v(x, z)^2} q(x, z, 0)$$


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whence

$$\tilde{F}_0[v]^* d(x, z) = \frac{2}{v(x, z)^2} q(x, z, 0)$$

Standard description of the SSR migration algorithm:

- downward continue data (i.e. solve for  $q$ )
- image at  $t = 0$ .

The art of SSR migration: computable approximations to  $b(x, z, D_x, D_t)$  - swimming pool operator, many successors.

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## Proof of the Claim

Unfinished business: proof of claim

Depends on celebrated **Propagation of Singularities** theorem of Hörmander (1970).

Given symbol  $p(\mathbf{x}, \boldsymbol{\xi})$ , order  $m$ , with asymptotic expansion, define *bicharacteristics* as solutions  $(\mathbf{x}(t), \boldsymbol{\xi}(t))$  of Hamiltonian system

$$\frac{d\mathbf{x}}{dt} = \frac{\partial p}{\partial \boldsymbol{\xi}}(\mathbf{x}, \boldsymbol{\xi}), \quad \frac{d\boldsymbol{\xi}}{dt} = -\frac{\partial p}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\xi})$$

with  $p(\mathbf{x}(t), \boldsymbol{\xi}(t)) \equiv 0$ .

**Theorem:** Suppose  $p(\mathbf{x}, D)u = f$ , and suppose that for  $t_0 \leq t \leq t_1$ ,  $(\mathbf{x}(t), \boldsymbol{\xi}(t)) \notin WF(f)$ . Then either  $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \leq t \leq t_1\} \subset WF(u)$  or  $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \leq t \leq t_1\} \subset T^*(\mathbf{R}^n) - WF(u)$ .

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P of S has at least two distinct proofs:

- Nirenberg, 1972
- Hörmander, 1970 (in Taylor, 1981)

Proof of claim: check that bicharacteristics for SSR operator are just upcoming rays of geom. optics for wave equation. These pass into  $t < 0$  where RHS is smooth, also initial condn at large  $z$  is smooth - so each ray has one “end” outside of  $WF(\tilde{w}_1)$ . If ray carries singularity, must pass of  $WF$  of  $w$ , but then it’s entirely contained by P of S applied to  $w$ . **q. e. d.**

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## Reverse Depth Migration, Prestack

Nonzero offset (“prestack”): starting point is integral representation of the scattered field

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int dx \frac{2r(\mathbf{x})}{v(\mathbf{x})^2} \int ds G(\mathbf{x}_r, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{x})$$

By analogy with zero offset case, would like to view this as “exploding reflectors in both directions”: reflectors propagate energy upward to sources and to receivers.

However can’t do this because reflection location is *same* for both.

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## The “survey sinking” idea

Bold stroke: introduce a new space variable  $\mathbf{y}$  (a “sunken source”, think of  $\mathbf{x}$  as a “sunken receiver”), define

$$\tilde{F}[v]R(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int \int dx dy R(\mathbf{x}, \mathbf{y}) \int ds G(\mathbf{x}_r, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{y})$$

and note that  $\tilde{F}[v]R = F[v]r$  if

$$R(\mathbf{x}, \mathbf{y}) = \frac{2r}{v^2} \left( \frac{\mathbf{x} + \mathbf{y}}{2} \right) \delta(\mathbf{x} - \mathbf{y})$$

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This trick decomposes  $F[v]$  into *two* “exploding reflectors”:

$$\tilde{F}[v]R(\mathbf{x}_r, t; \mathbf{x}_s) = u(\mathbf{x}, t; \mathbf{x}_s)|_{\mathbf{x}=\mathbf{x}_r}$$

where

$$\begin{aligned} \left( \frac{1}{v(\mathbf{x})^2} \frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{x}}^2 \right) u(\mathbf{x}, t; \mathbf{x}_s) &= \int dy R(\mathbf{x}, \mathbf{y}) G(\mathbf{x}_s, t; \mathbf{y}) \\ &\equiv w_s(\mathbf{x}_s, t; \mathbf{x}) \end{aligned}$$

(“upward continue the receivers”),

$$\left( \frac{1}{v(\mathbf{y})^2} \frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{y}}^2 \right) w_s(\mathbf{y}, t; \mathbf{x}) = R(\mathbf{x}, \mathbf{y}) \delta(t)$$

(“upward continue the sources”).

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This factorization of  $F[v]$  ( $r \mapsto R \mapsto \tilde{F}[v]R$ ) leads to a reverse time computation of adjoint  $\tilde{F}[v]^*$  - will discuss this later.

It's equally possible to continue the receivers first, then the sources, which leads to

$$\left( \frac{1}{v(\mathbf{y})^2} \frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{y}}^2 \right) u(\mathbf{x}_r, t; \mathbf{y}) = \int dx R(\mathbf{x}, \mathbf{y}) G(\mathbf{x}_r, t; \mathbf{x})$$

$$\equiv w_r(\mathbf{x}_r, t; \mathbf{y})$$

(“upward continue the sources”),

$$\left( \frac{1}{v(\mathbf{x})^2} \frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{x}}^2 \right) w_r(\mathbf{x}, t; \mathbf{y}) = R(\mathbf{x}, \mathbf{y}) \delta(t)$$

(“upward continue the receivers”).

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## The DSR Assumption

Apply reverse depth concept: as before, go 2D temporarily,  $\mathbf{x} = (x, z_r)$ ,  $\mathbf{y} = (y, z_s)$ , all sources and receivers on  $z = 0$ .

**Double Square Root** (“DSR”) assumption: For some  $\delta > 0$ , the wavefield  $u$  satisfies

$$(x, z_r, t, y, z_s, \xi, \zeta_s, \tau, \eta, \zeta_r) \in WF(u) \Rightarrow$$

$$(x, t, \xi, \tau) \in \mathcal{P}_\delta(z_r), \quad (y, t, \eta, \tau) \in \mathcal{P}_\delta(z_s), \quad \text{and } \zeta_r \tau > 0, \quad \zeta_s \tau > 0,$$

As for SSR, there is a ray-theoretic interpretation: rays from source and receiver to scattering point stay away from the vertical and decrease in  $z$  for increasing  $t$ , i.e. they are all upcoming.



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Since  $z$  will be singled out (and eventually  $R(\mathbf{x}, \mathbf{y})$  will have a factor of  $\delta(\mathbf{x}, \mathbf{y})$ ), impose the constraint that

$$R(x, z, x, z_s) = \tilde{R}(x, y, z) \delta(z - z_s)$$

Define upcoming projections as for SSR:

$$\tilde{w}_s = \left( \frac{\partial}{\partial z_s} + ib(y, z_s, D_y, D_t) \right) w_s,$$

$$\tilde{w}_r = \left( \frac{\partial}{\partial z_r} + ib(x, z_r, D_x, D_t) \right) w_r,$$

$$\tilde{u} = \left( \frac{\partial}{\partial z_s} + ib(y, z_s, D_y, D_t) \right) \left( \frac{\partial}{\partial z_r} + ib(x, z_r, D_x, D_t) \right) u$$


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Except for lower order commutators which we justify throwing away as before,

$$\left( \frac{\partial}{\partial z_s} - ib(y, z_s, D_y, D_t) \right) \tilde{w}_s = \tilde{R} \delta(z_r - z_s) \delta(t),$$

$$\left( \frac{\partial}{\partial z_r} - ib(x, z_r, D_x, D_t) \right) \tilde{w}_r = \tilde{R} \delta(z_r - z_s) \delta(t),$$

$$\left( \frac{\partial}{\partial z_r} - ib(x, z_r, D_x, D_t) \right) \tilde{u} = \tilde{w}_s$$

$$\left( \frac{\partial}{\partial z_s} - ib(y, z_s, D_y, D_t) \right) \tilde{u} = \tilde{w}_r$$

Initial (final) conditions are that  $\tilde{w}_r$ ,  $\tilde{w}_s$ , and  $\tilde{u}$  all vanish for large  $z$  - the equations are to be solve in decreasing  $z$  (“upward continuation”).

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Simultaneous upward continuation:

$$\begin{aligned}\frac{\partial}{\partial z}\tilde{u}(x, z, t; y, z) &= \frac{\partial}{\partial z_r}\tilde{u}(x, z_r, t; y, z)|_{z=z_r} + \frac{\partial}{\partial z_s}\tilde{u}(x, z, t; y, z_s)|_{z=z_s} \\ &= [ib(x, z_r, D_x, D_t)\tilde{u} + \tilde{w}_s + ib(y, z_s, D_y, D_t)\tilde{u} + \tilde{w}_r]_{z_r=z_s=z}\end{aligned}$$

Since  $\tilde{w}_s(y, z, t; x, z) = \tilde{w}_r(x, z, t; y, z) = \tilde{R}(x, y, z)\delta(t)$ ,  $\tilde{u}$  is seen to satisfy the

**DSR modeling equation:**

$$\left(\frac{\partial}{\partial z} - ib(x, z, D_x, D_t) - ib(y, z, D_y, D_t)\right)\tilde{u}(x, z, t; y, z) = 2\tilde{R}(x, y, z)\delta(t)$$

$$\tilde{F}[v]\tilde{R}(x_r, t; x_s) = \tilde{u}(x_r, 0, t; x_s, 0)$$

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## DSR Migration

Computation of adjoint follows same pattern as for SSR, and leads to

**DSR migration equation:** solve

$$\left( \frac{\partial}{\partial z} - ib(x, z, D_x, D_t) - ib(y, z, D_y, D_t) \right) \tilde{q}(x, y, z, t) = 0$$

in *increasing*  $z$  with initial condition at  $z = 0$ :

$$\tilde{q}(x_r, x_s, 0, t) = d(x_r, x_s, t)$$

Then  $\tilde{F}[v]^* d(x, y, z) = \tilde{q}(x, y, z, 0)$

The physical DSR model has  $\tilde{R}(x, y, z) = r(x, z)\delta(x - y)$ , so final step in DSR computation of  $F[v]^*$  is adjoint of  $r \mapsto \tilde{R}$ :

$$F[v]^* d(x, z) = \tilde{q}(x, x, z, 0)$$

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# Standard description of DSR migration

(See Claerbout, IEI):

- downward continue sources and receivers (solve DSR migration equation)
- image at  $t = 0$  and zero offset ( $x = y$ )

Another moniker: “survey sinking”: DSR field  $\tilde{q}$  is (related to) the field that you would get by conducting the survey with sources and receivers at depth  $z$ . At any given depth, the zero-offset, time-zero part of the field is the instantaneous response to scatterers on which source = receiver is sitting, therefore constitutes an image.

As for SSR, the art of DSR migration is in the approximation of the DSR operator.

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## Remarks

Stolk and deHoop (2001) derived DSR modeling and migration via a more systematic argument than that used here, involving  $\Psi$ DO matrix factorization of the wave equation written as a first order evolution system in  $z$ . This idea goes back to Taylor (1975) who used it to show that singularities propagating along bicharacteristics reflect as expected at boundaries.

Stolk (2003) has also carried out a very careful global construction of a family of SSR  $\Psi$ DOs which are of non-classical type at near-horizontal directions (“nearly evanescent waves”). This construction should lead to more reliable discretizations.

The last part of the course will present the various apparently ad-hoc “prestack modeling” ideas within a unified framework.