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# Mathematics of Seismic Imaging

## Part V

William W. Symes

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# A step beyond linearization: velocity analysis

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# Velocity Analysis

Partially linearized seismic inverse problem (“velocity analysis”): given observed seismic data  $d$ , find smooth *velocity*  $v \in \mathcal{E}(X)$ ,  $X \subset \mathbf{R}^3$  oscillatory *reflectivity*  $r \in \mathcal{E}'(X)$  so that

$$F[v]r \simeq d$$

Acoustic partially linearized model: acoustic potential field  $u$  and its perturbation  $\delta u$  solve

$$\left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) u = \delta(t) \delta(\mathbf{x} - \mathbf{x}_s), \quad \left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta u = 2r \nabla^2 u$$

plus suitable bdry and initial conditions.

$$F[v]r = \frac{\partial \delta u}{\partial t} \Big|_Y$$

*data acquisition manifold*  $Y = \{(\mathbf{x}_r, t; \mathbf{x}_s)\} \subset \mathbf{R}^7$ ,  $\dim Y \leq 5$  (many idealizations here!).

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$F[v] : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y)$  is a linear map (FIO of order 1), but dependence on  $v$  is quite nonlinear, so this inverse problem is nonlinear.

Agenda:

- reformulation of inverse problem via *extensions*
- “standard processing” extension and standard VA
- the surface oriented extension and standard MVA
- the  $\Psi$ DO property and why it’s important
- global failure of the  $\Psi$ DO property for the SOE
- Claerbout’s depth oriented extension has the  $\Psi$ DO property
- differential semblance

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## Extensions

*Extension* of  $F[v]$ : manifold  $\bar{X}$  and maps  $\chi : \mathcal{E}'(X) \rightarrow \mathcal{E}'(\bar{X})$ ,  $\bar{F}[v] : \mathcal{E}'(\bar{X}) \rightarrow \mathcal{D}'(Y)$  so that

$$\begin{array}{ccccc}
 & & \bar{F}[v] & & \\
 & \mathcal{E}'(\bar{X}) & \rightarrow & \mathcal{D}'(Y) & \\
 \chi \uparrow & & & \uparrow & \text{id} \\
 & \mathcal{E}'(X) & \rightarrow & \mathcal{D}'(Y) & \\
 & & F[v] & & 
 \end{array}$$

commutes.

*Invertible extension*:  $\bar{F}[v]$  has a *right parametrix*  $\bar{G}[v]$ , i.e.  $I - \bar{F}[v]\bar{G}[v]$  is smoothing. [The trivial extension -  $\bar{X} = X$ ,  $\bar{F} = F$  - is virtually never invertible.] Also  $\chi$  has a *left inverse*  $\eta$ .

Reformulation of inverse problem: given  $d$ , find  $v$  so that  $\bar{G}[v]d \in \mathcal{R}(\chi)$  (implicitly determines  $r$  also!).

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## Example 1: Standard VA extension

Treat each CMP *as if it were the result of an experiment performed over a layered medium*, but permit the layers to vary with midpoint.

Thus  $v = v(z)$ ,  $r = r(z)$  for purposes of analysis, but at the end  $v = v(\mathbf{x}_m, z)$ ,  $r = r(\mathbf{x}_m, z)$ .

$$F[v]R(\mathbf{x}_m, h, t) \simeq A(\mathbf{x}_m, h, z(\mathbf{x}_m, h, t))R(\mathbf{x}_m, z(\mathbf{x}_m, h, t))$$

Here  $z(\mathbf{x}_m, h, t)$  is the inverse of the 2-way traveltime

$$t(\mathbf{x}_m, h, z) = 2\tau(\mathbf{x}_m + (h, 0, z), \mathbf{x}_m)_{v=v(\mathbf{x}_m, z)}$$

computed with the layered velocity  $v(\mathbf{x}_m, z)$ , i.e.

$$z(\mathbf{x}_m, h, t(\mathbf{x}_m, h, z')) = z'.$$

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That is,  $F[v]$  is a change of variable followed by multiplication by a smooth function. NB: industry standard practice is to use vertical traveltime  $t_0$  instead of  $z$  for depth variable.

Can write this as  $F[v] = \bar{F}S^*$ , where  $\bar{F}[v] = N[v]^{-1}M[v]$  has right parametrix  $\bar{G}[v] = M[v]N[v]$ :

$N[v] = \mathbf{NMO\ operator}$   $N[v]d(\mathbf{x}_m, h, z) = d(\mathbf{x}_m, h, t(\mathbf{x}_m, h, z))$

$M[v] = \text{multiplication by } A$

$S = \mathbf{stacking\ operator}$

$$Sf(\mathbf{x}_m, z) = \int dh f(\mathbf{x}_m, h, z), \quad S^*r(\mathbf{x}_m, h, z) = r(\mathbf{x}, z)$$


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Identify as extension:  $\bar{F}[v], \bar{G}[v]$  as above,  $X = \{\mathbf{x}_m, z\}, H = \{h\}, \bar{X} = X \times H, \chi = S^*, \eta = S$  - the invertible extension properties are clear.

Standard names for the Standard VA extension objects:  $\bar{F}[v]$  = “inverse NMO”,  $\bar{G}[v]$  = “NMO” [often the multiplication op  $M[v]$  is neglected];  $\eta$  = “stack”,  $\chi$  = “spread”

**How this is used for velocity analysis:** Look for  $v$  that makes  $\bar{G}[v]d \in \mathcal{R}(\chi)$

So what is  $\mathcal{R}(\chi)$ ?  $\chi[r](\mathbf{x}_m, z, h) = r(\mathbf{x}_m, z)$  Anything in range of  $\chi$  is *independent of*  $h$ . Practical issues  $\Rightarrow$  replace “independent of” with “smooth in”.



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## Flatten them gathers!

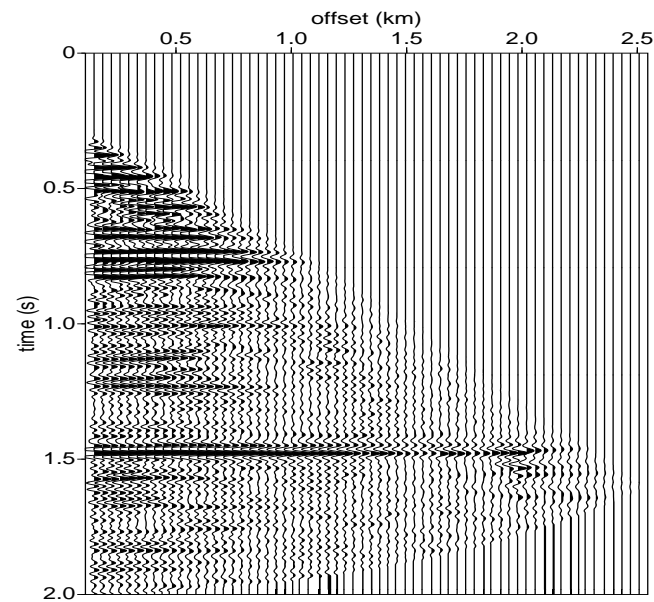
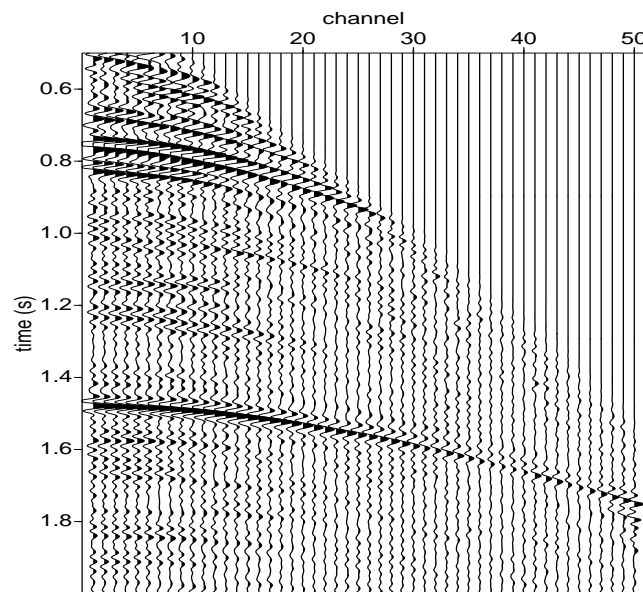
Inverse problem reduced to: adjust  $v$  to make  $\bar{G}[v]d^{\text{obs}}$  smooth in  $h$ , i.e. *flat* in  $z, h$  display for each  $\mathbf{x}_m$  (*NMO-corrected CMP*).

Replace  $z$  with  $t_0$ ,  $v$  with  $v_{\text{RMS}}$  em localizes computation: reflection through  $\mathbf{x}_m, t_0, 0$  *flattened* by adjusting  $v_{\text{RMS}}(\mathbf{x}_m, t_0) \Rightarrow$  1D search, do by visual inspection.

Various aids - NMO corrected CMP gathers, velocity spectra, etc.

See: Claerbout: *Imaging the Earth's Interior*

WWS: MGSS 2000 notes



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**Left:** part of survey ( $d$ ) from North Sea (thanks: Shell Research), lightly preprocessed.

**Right:** restriction of  $\bar{G}[v]d$  to  $\mathbf{x}_m = \text{const}$  (function of depth, offset): shows rel. sm'ness in  $h$  (offset) for properly chosen  $v$ .

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## Example 2: Surface oriented or standard MVA extension

. This only works where Earth is “nearly layered”. Where this fails, replace NMO by prestack migration.

Shot version:  $\Sigma_s$  = set of shot locations,  $\bar{X} = X \times \Sigma_s$ ,  $\chi[r](\mathbf{x}, \mathbf{x}_s) = r(\mathbf{x})$ .

$$\bar{F}[v]\bar{r}(\mathbf{x}_r, t, \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int dx \bar{r}(\mathbf{x}, \mathbf{x}_s) \int ds G(\mathbf{x}_r, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{x})$$

Offset version (preferred because it minimizes truncation artifacts):  $\Sigma_h$  = set of half-offsets in data,  $\bar{X} = X \times \Sigma_h$ ,  $\chi[r](\mathbf{x}, \mathbf{h}) = r(\mathbf{x})$ .

$$\bar{F}[v]\bar{r}(\mathbf{x}_s, t, \mathbf{h}) = \frac{\partial^2}{\partial t^2} \int dx \bar{r}(\mathbf{x}, \mathbf{h}) \int ds G(\mathbf{x}_s + \mathbf{h}, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{x})$$

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[Parametrize data with source location  $\mathbf{x}_s$ , time  $t$ , offset  $\mathbf{h}$ .] **NB:** note that both versions are “block diagonal” - family of operators (FIOs) parametrized by  $\mathbf{x}_s$  or  $\mathbf{h}$ .

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# Properties of SOE

Beylkin (1985), Rakesh (1988): if  $\|v\|_{C^2(X)}$  “not too big”, then

- $\bar{F}$  has **the  $\Psi$ DO property**:  $\bar{F}\bar{F}^*$  is  $\Psi$ DO
- singularities of  $\bar{F}\bar{F}^*d \subset$  singularities of  $d$
- straightforward construction of right parametrix  $\bar{G} = \bar{F}^*Q$ ,  $Q = \Psi$ DO, also as generalized Radon Transform - explicitly computable.

Range of  $\chi$  (offset version):  $\bar{r}(\mathbf{x}, \mathbf{h})$  independent of  $\mathbf{h} \Rightarrow$  “semblance principle”: find  $v$  so that  $\bar{G}[v]d^{\text{obs}}$  is independent of  $\mathbf{h}$ . Practical limitations  $\Rightarrow$  replace “independent of  $\mathbf{h}$ ” by “smooth in  $\mathbf{h}$ ”.

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# Industrial MVA

Application of these ideas = industrial practice of migration velocity analysis.

Idea: twiddle  $v$  until  $\bar{G}[v]d^{\text{obs}}$  is smooth in  $h$ .

Since it is hard to inspect  $\bar{G}[v]d^{\text{obs}}(x, y, z, h)$ , pull out subset for constant  $x, y =$  **common image gather** (“CIG”): display function of  $z, h$  for fixed  $x, y$ . These play same role as NMO corrected CMP gathers in layered case.

Try to adjust  $v$  so that selected CIGs are *flat* - just as in Standard VA. This is much harder, as there is no RMS velocity trick to localize the computation - each CIG depends globally on  $v$ .

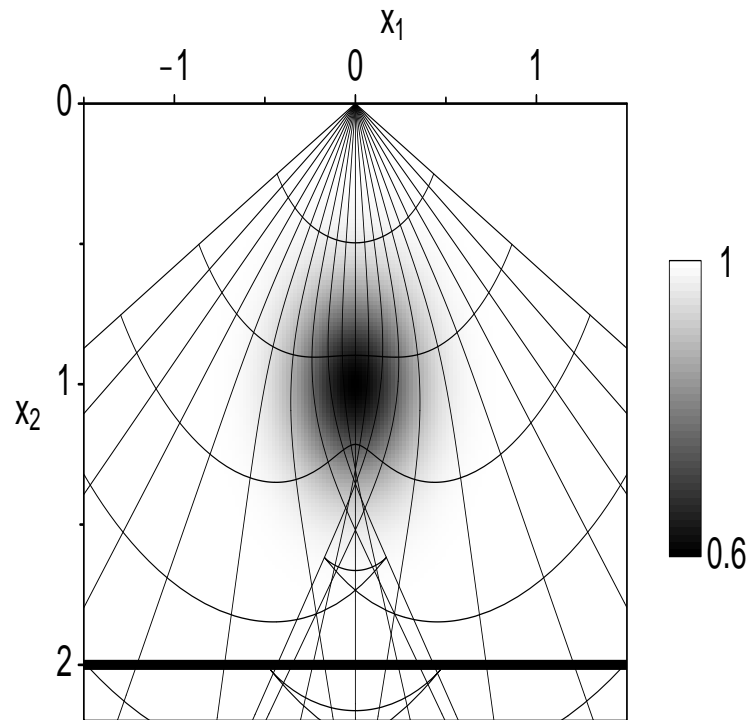
Description, some examples: Yilmaz, *Seismic Data Processing*.

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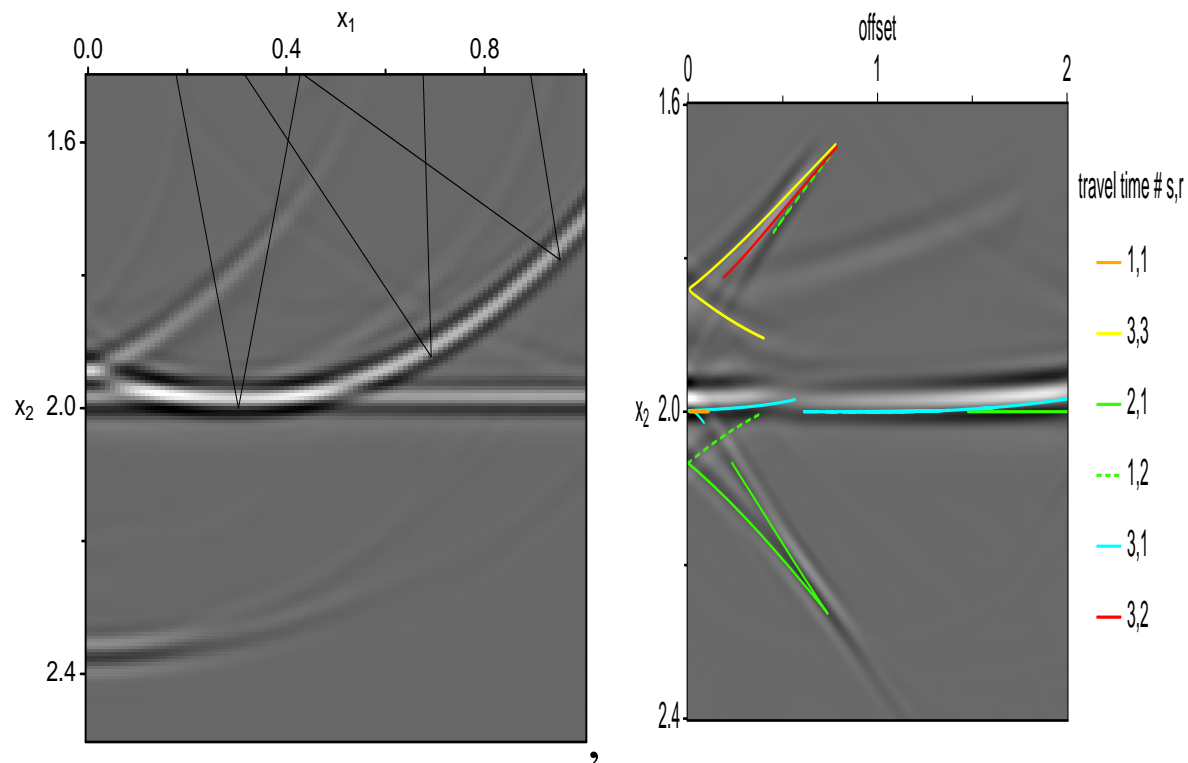
## Bad news

Nolan (1997): big trouble! In general, standard extension does **not** have the  $\Psi$ DO property. Geometric optics analysis: for  $\|v\|_{C^2(X)}$  “large”, multiple rays connect source, receiver to reflecting points in  $X$ ; block diagonal structure of  $\bar{F}[v] \Rightarrow$  info necessary to distinguish multiple rays is *projected out*.



Example (Stolk & WWS, 2001): Gaussian lens over flat reflector at depth  $z$  ( $r(\mathbf{x}) = \delta(x_1 - z)$ ,  $x_1 = \text{depth}$ ).





**Left:** Const.  $h$  slice of  $\bar{G}d$ : several refl. points corresponding to same singularity in  $d^{\text{obs}}$ .

**Right:** CIG (const.  $x, y$  slice) of  $\bar{G}d$ : not smooth in  $h$ !

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## Example 3: Claerbout's depth oriented extension

Standard MVA extension only works when Earth has simple ray geometry. Claerbout proposed alternative extension:

$\Sigma_d$  = somewhat arbitrary set of vectors near 0 (“offsets”),  $\bar{X} = X \times \Sigma_d$ ,  $\chi[r](\mathbf{x}, \mathbf{h}) = r(\mathbf{x})\delta(\mathbf{h})$ ,  $\eta[\bar{r}](\mathbf{x}) = \bar{r}(\mathbf{x}, 0)$

$$\begin{aligned}\bar{F}[v]\bar{r}(\mathbf{x}_s, t, \mathbf{x}_r) &= \frac{\partial^2}{\partial t^2} \int dx \int_{\Sigma_d} dh \bar{r}(\mathbf{x}, \mathbf{h}) \int ds G(\mathbf{x}_s, t - s; \mathbf{x} + 2\mathbf{h}) G(\mathbf{x}_r, s; \mathbf{x}) \\ &= \frac{\partial^2}{\partial t^2} \int dx \int_{\mathbf{x}+2\Sigma_d} dy \bar{r}(\mathbf{x}, \mathbf{y} - \mathbf{x}) \int ds G(\mathbf{x}_s, t - s; \mathbf{y}) G(\mathbf{x}_r, s; \mathbf{x})\end{aligned}$$

**NB:** in this formulation, there appears to be too many model parameters.

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# Shot record modeling

for each  $\mathbf{x}_s$  solve

$$\bar{F}[v]\bar{r}(\mathbf{x}_r, t; \mathbf{x}_s) = u(\mathbf{x}, t; \mathbf{x}_s)|_{\mathbf{x}=\mathbf{x}_r}$$

where

$$\left( \frac{1}{v(\mathbf{x})^2} \frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{x}}^2 \right) u(\mathbf{x}, t; \mathbf{x}_s) = \int_{\mathbf{x}+2\Sigma_d} dy \bar{r}(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, t; \mathbf{x}_s)$$

$$\left( \frac{1}{v(\mathbf{y})^2} \frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{y}}^2 \right) G(\mathbf{y}, t; \mathbf{x}_s) = \delta(t) \delta(\mathbf{x}_s - \mathbf{y})$$

Finite difference scheme: form RHS for eqn 1, step  $u$ ,  $G$  forward in  $t$ .

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## Computing $\bar{G}[v]$

Instead of parametrix, be satisfied with adjoint.

Reverse time adjoint computation - specify adjoint field as in standard reverse time prestack migration:

$$\left( \frac{1}{v(\mathbf{x})^2} \frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{x}}^2 \right) w(\mathbf{x}, t; \mathbf{x}_s) = \int dx_r d(\mathbf{x}_r, t; \mathbf{x}_s) \delta(\mathbf{x} - \mathbf{x}_r)$$

with  $w(\mathbf{x}, t; \mathbf{x}_s) = 0, t \gg 0$ . Then

$$\bar{F}[v]^* d(\mathbf{x}, \mathbf{h}) = \int dx_s \int dt G(\mathbf{x} + 2\mathbf{h}, t; \mathbf{x}_s) w(\mathbf{x}, t; \mathbf{x}_s)$$

i.e. exactly the same computation as for reverse time prestack, except that crosscorrelation occurs at an offset  $2\mathbf{h}$ .

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## Nomenclature

NB: the “usual computation” of  $\bar{G}[v]$  is either DSR or a variant of shot record computation of previous slide using depth extrapolation.  $\mathbf{h}$  is usually restricted to be horizontal, i.e.  $h_3 = 0$ .

Common names: shot-geophone or survey-sinking migration (with DSR), or shot record migration.

“Downward continue sources and receivers, image at  $t = 0, h = 0$ ”

These are what is typically meant by “wave equation migration”!

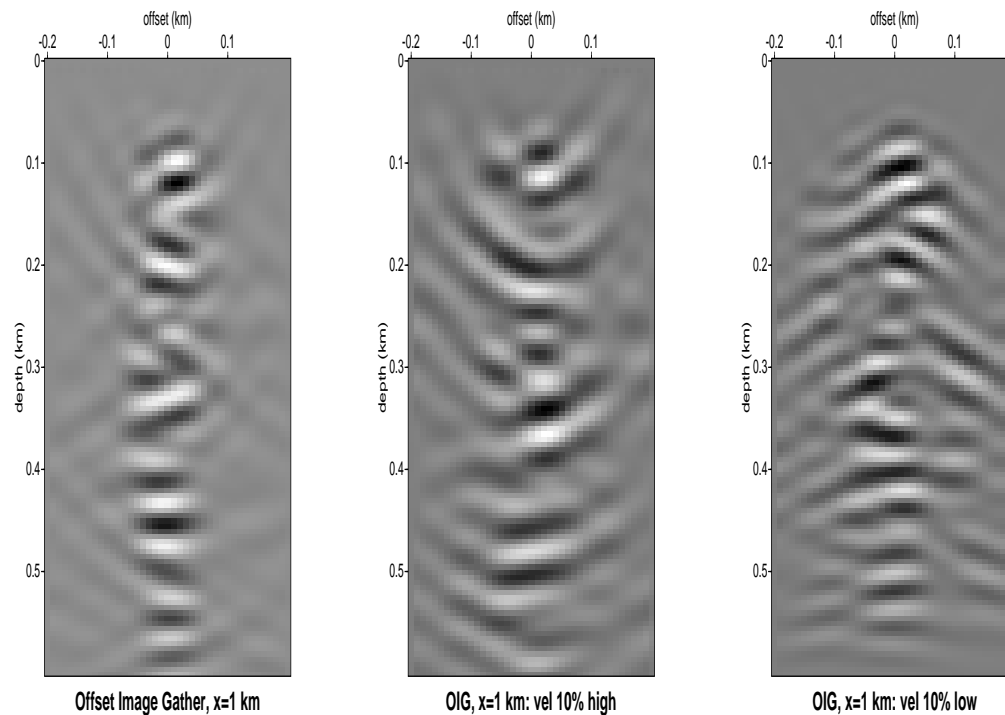
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What should be the character of the image when the velocity is correct?

Hint: for simulation of seismograms, the input reflectivity had the form  $r(\mathbf{x})\delta(\mathbf{h})$ .

Therefore guess that when velocity is correct, *image is concentrated near  $h = 0$* .

Examples: 2D finite difference implementation of reverse time method. Correct velocity  $\equiv 1$ . Input reflectivity used to generate synthetic data: random! For output reflectivity (image of  $\bar{F}[v]^*$ ), constrain offset to be horizontal:  $\bar{r}(\mathbf{x}, \mathbf{h}) = \tilde{r}(\mathbf{x}, h_1)\delta(h_3)$ . Display CIGs (i.e.  $x_1 = \text{const.}$  slices).



Two way reverse time horizontal offset S-G image gathers of data from random reflectivity, constant velocity. From left to right: correct velocity, 10% high, 10% low.

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## Stolk and deHoop, 2001

Claerbout extension has the  $\Psi$ DO

property, at least when restricted to  $\bar{r}$  of the form  $\bar{r}(\mathbf{x}, \mathbf{h}) = R(\mathbf{x}, h_1, h_2)\delta(h_3)$ , and under DSR assumption.

Sketch of proof (after Rakesh, 1988):

This will follow from *injectivity* of wavefront or *canonical relation*  $C_{\bar{F}} \subset T^*(\bar{X}) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$  which describes singularity mapping properties of  $\bar{F}$ :

$$(\mathbf{x}, \mathbf{h}, \xi, \nu, \mathbf{y}, \eta) \in C_{F_\delta[v]} \Leftrightarrow$$

for some  $u \in \mathcal{E}'(\bar{X})$ ,  $(\mathbf{x}, \mathbf{h}, \xi, \nu) \in WF(u)$ , and  $(\mathbf{y}, \eta) \in WF(\bar{F}u)$



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## Characterization of $C_{\bar{F}}$

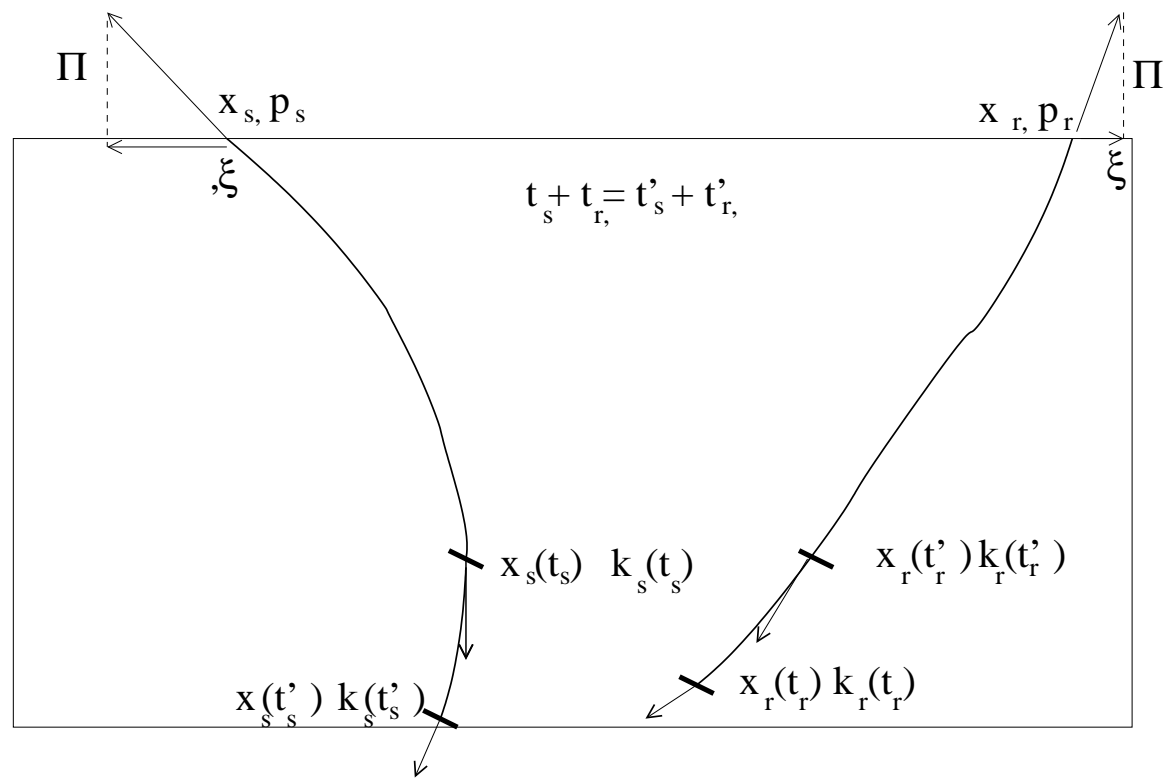
$$((\mathbf{x}, \mathbf{h}, \xi, \nu), (\mathbf{x}_s, t, \mathbf{x}_r, \xi_s, \tau, \xi_r)) \in C_{\bar{F}}[v] \subset T^*(\bar{X}) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$$

$\Leftrightarrow$  there are *rays of geometric optics*  $(\mathbf{X}_s, \Xi_s), (\mathbf{X}_r, \Xi_r)$  and times  $t_s, t_r$  so that

$$\Pi(\mathbf{X}_s(0), t, \mathbf{X}_r(0), \Xi_s(0), \tau, \Xi_r(0)) = (\mathbf{x}_s, t, \mathbf{x}_r, \xi_s, \tau, \xi_r),$$

$$\mathbf{X}_s(t_s) = \mathbf{x}, \mathbf{X}_r(t_r) = \mathbf{x} + 2\mathbf{h}, t_s + t_r = t,$$

$$\Xi_s(t_s) + \Xi_r(t_r) \parallel \xi, \Xi_s(t_s) - \Xi_r(t_r) \parallel \nu$$



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# Proof

Uses wave equations for  $u$ ,  $G$  and

- Gabor calculus: computes wave front sets of products, pullbacks, integrals, etc.  
See Duistermaat, Ch. 1.
- Propagation of Singularities Theorem

and that's all! [No integral representations, phase functions,...]

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Note intrinsic ambiguity: if you have a ray pair, move times  $t_s, t_r$  resp.  $t'_s, t'_r$ , for which  $t_s + t_r = t'_s + t'_r = t$  then you can construct two points  $(\mathbf{x}, \mathbf{h}, \boldsymbol{\xi}, \nu), (\mathbf{x}', \mathbf{h}', \boldsymbol{\xi}', \nu')$  which are candidates for membership in  $WF(\bar{r})$  and which satisfy the above relations with the same point in the cotangent bundle of  $T^*(Y)$ .

No wonder - there are too many model parameters!

Stolk and deHoop fix this ambiguity by imposing two constraints:

- DSR assumption: all rays carrying significant reflected energy (source or receiver) are upcoming.
- Restrict  $\bar{F}$  to the domain  $\mathcal{Z} \subset \mathcal{E}'(\bar{X})$

$$\bar{r} \in \mathcal{Z} \Leftrightarrow \bar{r}(\mathbf{x}, \mathbf{h}) = R(\mathbf{x}, h_1, h_2) \delta(h_3)$$


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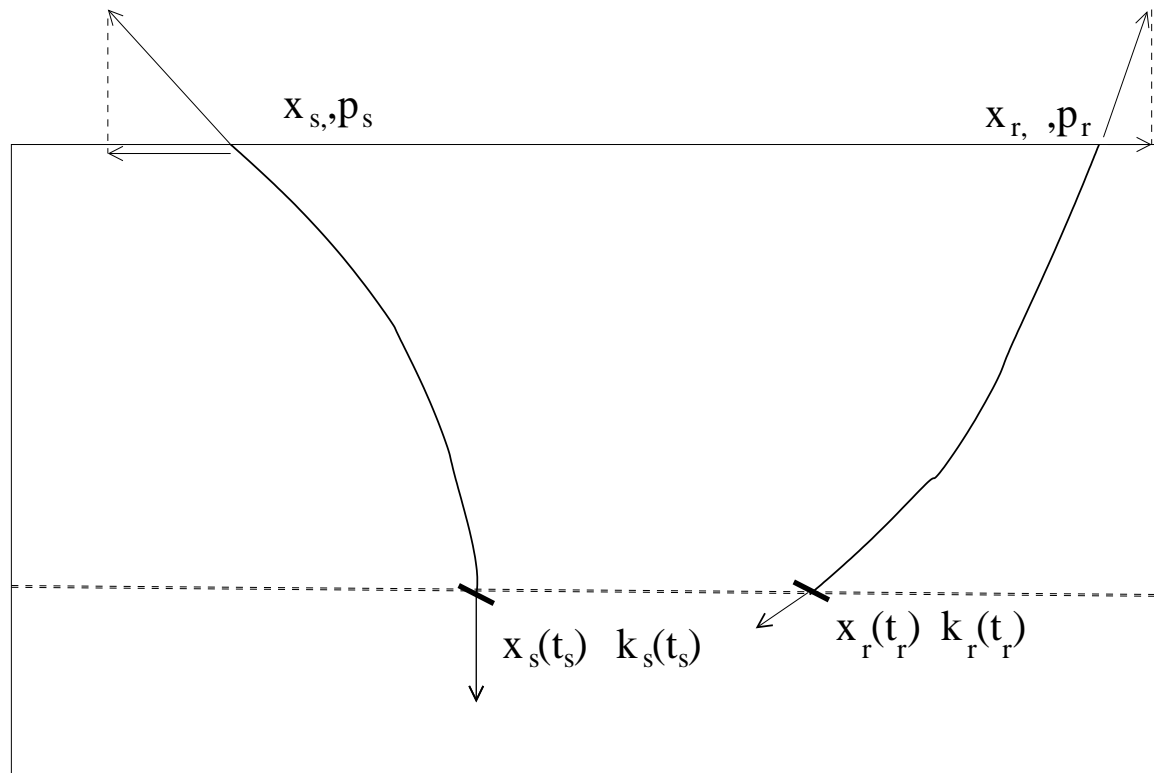
If  $\bar{r} \in \mathcal{Z}$ , then  $(\mathbf{x}, \mathbf{h}, \boldsymbol{\xi}, \nu) \in WF(\bar{r}) \Rightarrow h_3 = 0$ . So source and receiver rays in  $C_{\bar{F}}$  must terminate at same depth, to hit such a point.

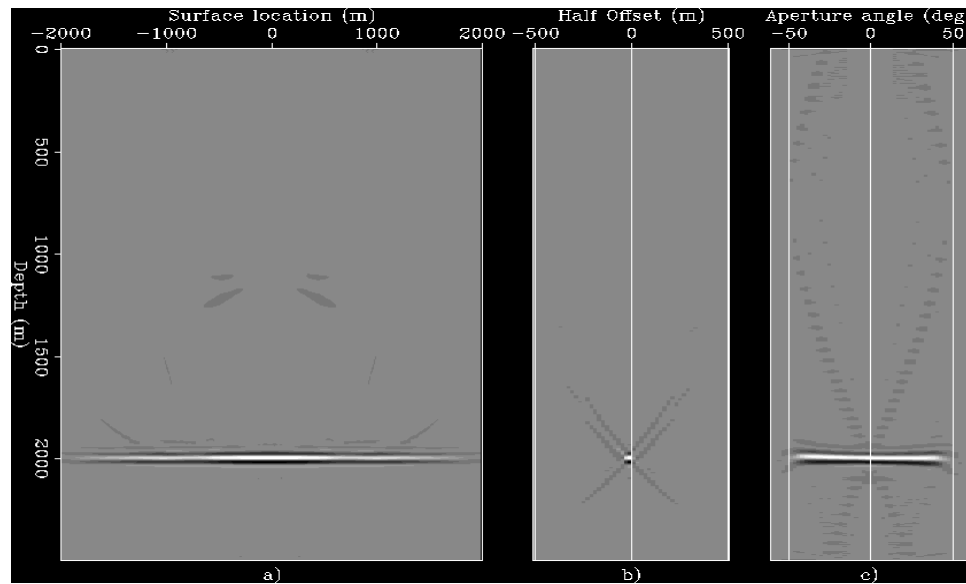
Because of DSR assumption, this fixes the traveltimes  $t_s, t_r$ .

**Restricted to  $\mathcal{Z}$ ,  $C_{\bar{F}}$  is injective.**

$$\Rightarrow C_{\bar{F}^* \bar{F}} = I$$

$\Rightarrow \bar{F}^* \bar{F}$  is  $\Psi$ DO when restricted to  $\mathcal{Z}$ .





Lens data, shot-geophone migration [B. Biondi, 2002]

Left: Image via DSR. Middle:  $\bar{G}[v]d$  - well-focused (at  $h = 0$ ), i.e. in range of  $\chi$  to extent possible. Right: Angle CIG.

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## Quantitative VA

Suppose  $W : \mathcal{E}'(\bar{X}) \rightarrow \mathcal{D}'(Z)$  annihilates range of  $\chi$ :

$$\mathcal{E}'(X) \xrightarrow{\chi} \mathcal{E}'(\bar{X}) \xrightarrow{W} \mathcal{D}'(Z) \rightarrow 0$$

and moreover  $W$  is bounded on  $L^2(\bar{X})$ . Then

$$J[v; d] = \frac{1}{2} \|W\bar{G}[v]d\|^2$$

*minimized* when  $[v, \eta\bar{G}[v]d]$  solves partially linearized inverse problem.

Construction of *annihilator* of  $\mathcal{R}(F[v])$  (Guillemin, 1985):

$$d \in \mathcal{R}(F[v]) \Leftrightarrow \bar{G}[v]d \in \mathcal{R}(\chi) \Leftrightarrow W\bar{G}[v]d = 0$$



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## Annihilators, annihilators everywhere...

For Standard Extended Model, several popular choices:

- $W = (I - \Delta)^{-\frac{1}{2}} \nabla_h$  (“differential semblance” - WWS, 1986)
- $W = I - \frac{1}{|H|} \int dh$  (“stack power” - Toldi, 1985)
- $W = I - \chi F[v]^\dagger \bar{F}[v] \Rightarrow$  minimizing  $J[v, d]$  equivalent to least squares.

For Claerbout extension, differential semblance  $W = h$ .

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## But not many are good for much...

Since *problem is huge*, only  $W$  giving rise to differentiable  $v \mapsto J[v, d]$  are useful - must be able to use Newton!!! Once again, idealize  $w(t) = \delta(t)$ .

**Theorem** (Stolk & WWS, 2003):  $v \mapsto J[v, d]$  smooth  $\Leftrightarrow W$  pseudodifferential.

i.e. only *differential semblance* gives rise to smooth optimization problem, *uniformly in source bandwidth*.

NB: Least squares embedded in larger family of optimization formulations, some (others) of which are tractable.

Numerical examples using synthetic and field data: WWS et al., Chauris & Noble 2001, Mulder & tenKroode 2002. deHoop et al. 2004.

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## Beyond Born

Nonlinear effects not included in linearized model: *multiple reflections*. Conventional approach: treat as *coherent noise*, attempt to eliminate - active area of research going back 40+ years, with recent important developments.

Why not model this “noise”?

Proposal: *nonlinear extensions* with  $F[v]r$  replaced by  $\mathcal{F}[c]$ . Create annihilators in same way (now also nonlinear), optimize differential semblance.

Nonlinear analog of Standard Extended Model appears to be *invertible* - in fact extended nonlinear inverse problem is *underdetermined*.

Open problems: no theory. Also must determine  $w(t)$  (Lailly SEG 2003).

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## And so on...

- Elasticity: theory of asymptotic Born inversion at smooth background in good shape (Beylkin & Burridge 1988, deHoop & Bleistein 1997). Theory of extensions, annihilators, differential semblance partially complete (Brandsberg-Dahl et al 2003).
- Anisotropy - work of deHoop (Brandsberg-Dahl et al 2003).
- Anelasticity - in the sedimentary section,  $Q = 100 - 1000$ , lower in gassy sediments and near surface. No mathematical results, but some numerics - Minkoff & WWS 1997, Blanch et al 1998.
- Source determination - actually always an issue. Some success in casting as an inverse problem - Minkoff & WWS 1997, Routh et al SEG 2003.
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## Selected References

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Cohen, J. K. and Bleistein, N.: An inverse method for determining small variations in propagation speed, *SIAM J. Appl. Math.* 32, 1977, pp. 784-799.

Rakesh: A linearized inverse problem for the wave equation, *Comm. PDE* 13, 1988, pp. 573-601.

Beylkin, G.: Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform, *J. Math. Phys* 26, 1985, pp. 99-108.

Ten Kroode, A. P. E., Smit, D. J., and Verdel, A. R.: A microlocal analysis of migration *Wave Motion* 28, 1998, pp. 149-172.

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deHoop, M. and Bleistein, N.: Generalized Radon Transform inversions for reflectivity in anisotropic elastic media, *Inverse Problems* 16, 1998, pp. 669-690.

Brandsberg-Dahl, S., De Hoop, M., and Ursin, B.: Focussing in dip and AVA compensation on scattering angle/azimuth common image gathers, *Geophysics* 68, 2003, pp. 232-254.

Stolk, C. and Symes, W. W.: Smooth objective functionals for seismic velocity inversion, *Inverse Problems* 19, 2003, pp. 73-89.

Stolk, C. and Symes, W. W.: Kinematic artifacts in prestack depth migration, *Geophysics* 68, May-June 2004.

Minkoff, S. E. and Symes, W. W.: Full waveform inversion of marine reflection data in the plane wave domain, *Geophysics* 62, 1997, pp. 540-553.

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Grau, G. and Versteeg, R.: *The Marmousi Experience: Practical Aspects of Inversion*, IFP-Technip, Paris, 1991.

Lions, J. L.: *Nonhomogeneous Boundary Value Problems and Applications*, Springer Verlag, Berlin, 1972.

Operto, S., Xu, S. and Lambaré, G.: Can we image quantitatively complex models with rays?, *Geophysics* 65, 2001, pp. 1223-1238.

Nolan, C. J. and Symes, W. W.: Global solution of a linearized inverse problem for the wave equation, *Comm. PDE* 22, 1997, pp. 919-952.

Stolk, C.: Microlocal analysis of a seismic linearized inverse problem, *Wave Motion* 32, 2000, pp. 267-290.

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Stolk, C. and deHoop, M.: Seismic inverse scattering in the wave equation approach, MSRI preprint, 2001.

Hagedoorn, F.: A process of seismic reflection interpretation, *Geophysical Prospecting*, 1954, pp. 85-127.

Duistermaat, J.: *Fourier Integral Operators*, Birkhauser, NY, 1996.

Taylor, M.: *Pseudodifferential Operators*, Princeton University Press, 1981.

Hörmander, L.: *The Analysis of Linear Partial Differential Operators*, vols. I-IV, Springer Verlag, 1983.

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Nirenberg, L.: *Lectures on Partial Differential Equations*, CMBS Lecture Notes, AMS, 1972.

Treves, F.: *Introduction To Pseudodifferential Operators And Fourier Integral Operators*, Plenum, 1980.

Claerbout, J.: *Basic Earth Imaging* (“BEI”) and *Imaging the Earth’s Interior* (“IEI”), available through SEP web site, [sepwww.stanford.edu](http://sepwww.stanford.edu).

Miller, D., Oristaglio, M. and Beylkin, G.: A new slant on seismic imaging, *Geophysics* 52, pp. 943-964, 1987.

Bleistein, N., Cohen, J. and Stockwell, J.: *The Mathematics of Seismic Imaging, Migration, and Inversion*, Springer, 2000.

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Dobrin, M. and Savit, C.: *Introduction to Geophysical Prospecting*

Yilmaz, O.: *Seismic Data Processing*, 2nd edition, SEG, 1999.

Symes, W.: *Mathematics of Reflection Seismology*, Lecture Notes,  
`www.trip.caam.rice.edu`, 1995.

Symes, W.: 2002 CIME Notes, `www.trip.caam.rice.edu`.