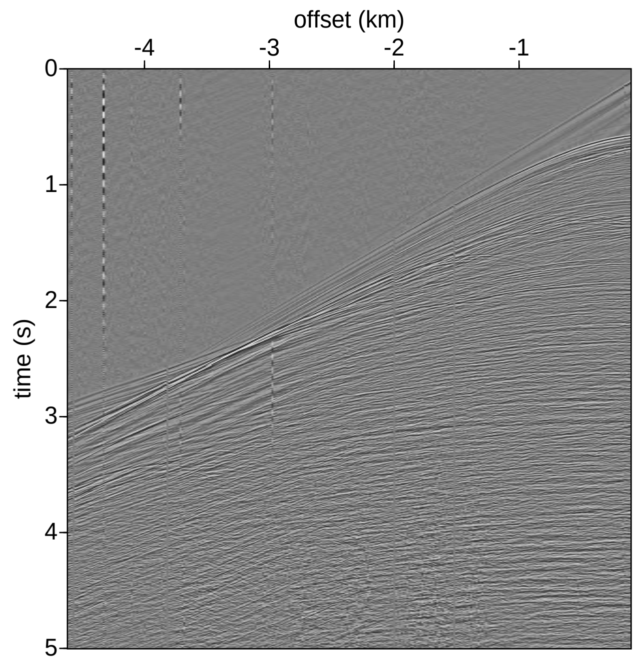
Mathematics of Seismic Imaging
Part I

William W. Symes

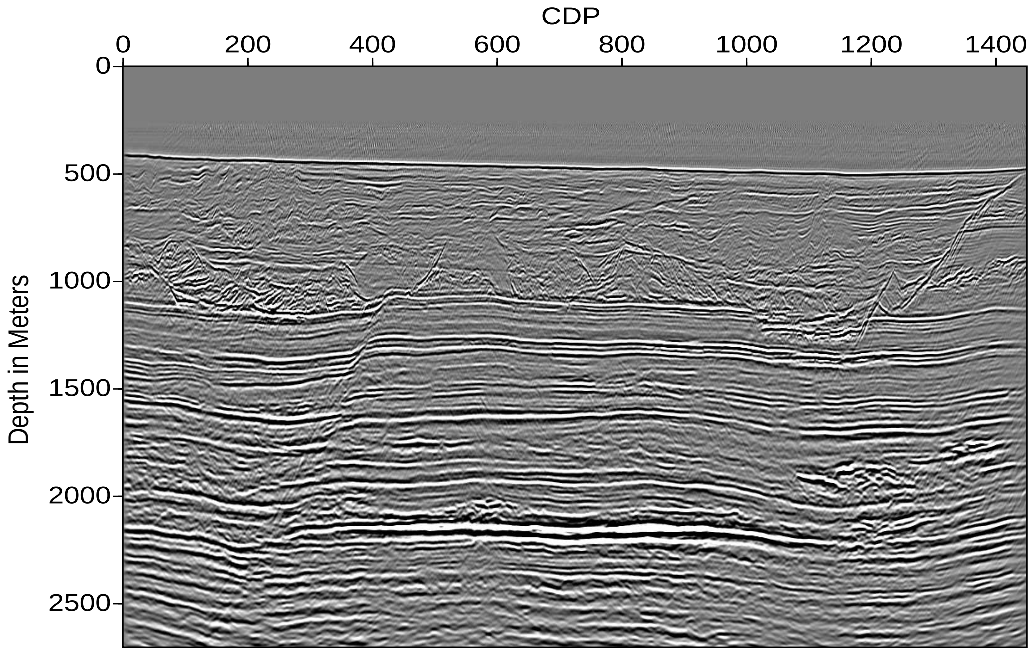
ExxonMobil R & E, June 2004

How do you turn lots of this...



(field seismogram from the Gulf of Mexico - thanks: Exxon.)

into this?



CDP
0
200
400
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1400
Depth in Meters
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2000
2500

Main Theme

Estimating the index of refraction (wave velocity) is *the central issue* in seismic imaging.

Combines elements of

- optics, radar, sonar - reflected wave imaging
- tomography - with curved rays

Many unanswered mathematical questions with practical implications!

A mathematical view

...of reflection seismic imaging, as practiced in the petroleum industry:

- an inverse problem, based on a model of seismic wave propagation
- contemporary practice relies on *partial linearization* and high-frequency asymptotics
- recent progress in understanding capabilities, limitations of methods based on linearization/asymptotics in presence of *strong refraction*: applications of *microlocal analysis* with implications for practice
- limitations of linearization lead to many open problems

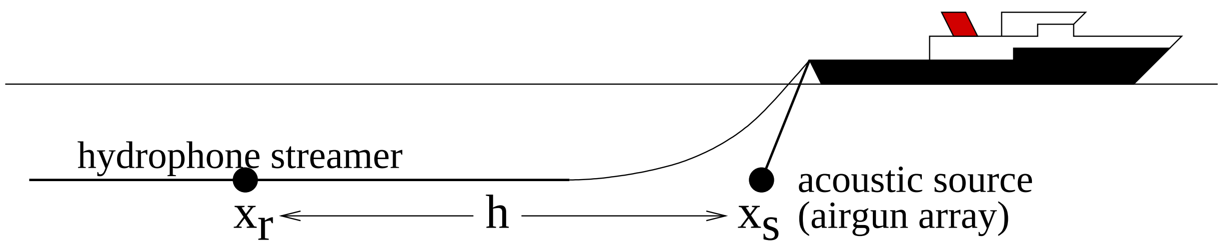
Agenda

1. Seismic inverse problem in the acoustic model: nature of data and model, linearization, reflectors and reflections idealized via *harmonic analysis of singularities*.
2. High frequency asymptotics: why adjoints of modeling operators are imaging operators (“Kirchhoff migration”). Beylkin theory of high frequency asymptotic inversion.
3. Adjoint state imaging with the wave equation: reverse time and reverse depth.
4. Geometric optics, Rakesh’s construction, and asymptotic inversion w/ caustics and multipathing, imaging artifacts, and prestack migration après Claerbout.
5. A step beyond linearization: a mathematical framework for velocity analysis.

1. The Acoustic Model and Linearization

Marine reflection seismology

- acoustic source (airgun array, explosives,...)
- acoustic receivers (hydrophone streamer, ocean bottom cable,...)
- recording and onboard processing



Land acquisition similar, but acquisition and processing are more complex. Vast bulk (90%+) of data acquired each year is marine.

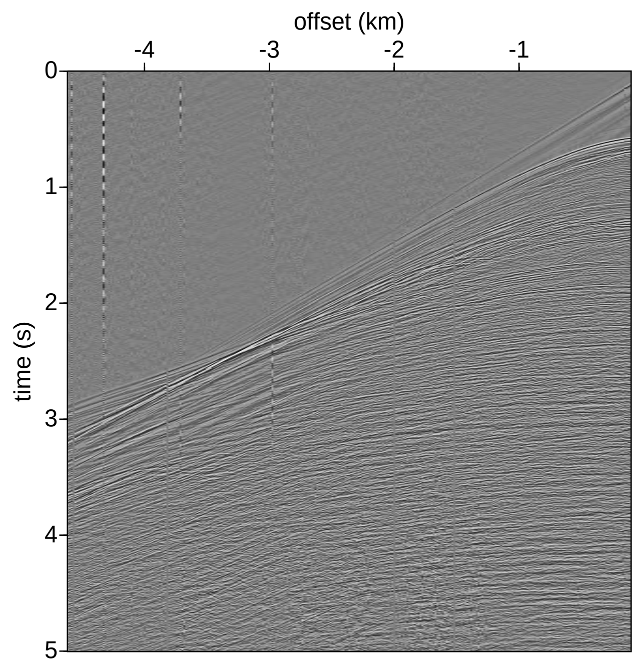
Data parameters: time t, source location $\mathbf{x}_s$, and receiver location $\mathbf{x}_r$ or *half offset* $\mathbf{h} = \frac{\mathbf{x}_r - \mathbf{x}_s}{2}$, $h = |\mathbf{h}|$.

Idealized marine "streamer" geometry: $\mathbf{x}_s$ and $\mathbf{x}_r$ lie roughly on constant depth plane, source-receiver lines are parallel $\rightarrow$ 3 spatial degrees of freedom (eg. $\mathbf{x}_s, h$): *codimension 1*. [Other geometries are interesting, eg. ocean bottom cables, but streamer surveys still prevalent.]

How much data? Contemporary surveys may feature

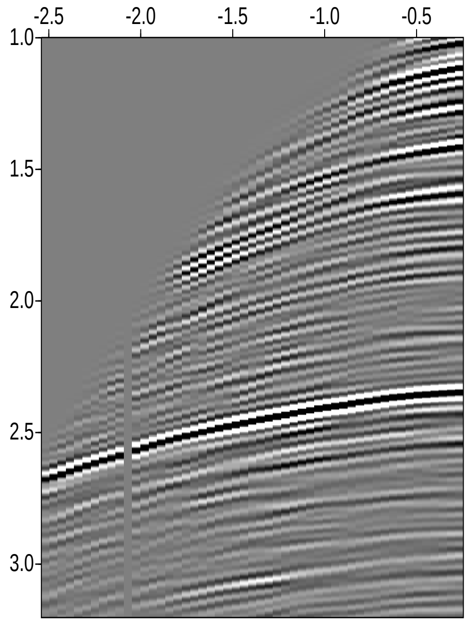
- Simultaneous recording by multiple streamers (up to 12!)
- Many (roughly) parallel ship tracks ("lines"), areal coverage
- single line ("2D") $\sim$ Gbyte; multiple lines ("3D") $\sim$ Tbyte

Shot gather, Mississippi Canyon



(thanks: Exxon)

Lightly processed...



bandpass filter 4-10-25-40 Hz, mute

Gathers: distinguished data subsets

Aka "bins", extracted from data after acquisition.

Characterized by common value of an acquisition parameter

- shot (or common source) gather: traces with same shot location $\mathbf{x}_s$ (previous expls)
- offset (or common offset) gather: traces with same half offset h
- ...

A key observation

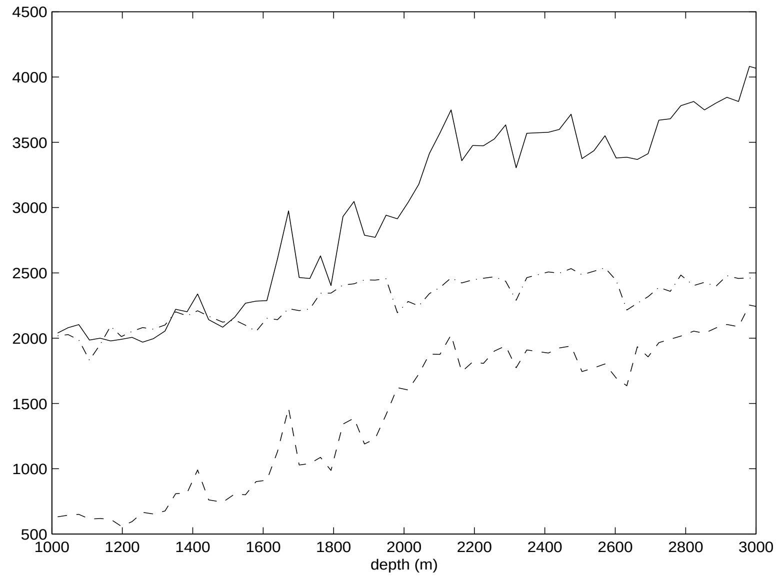
The most striking visual characteristic of seismic reflection data: presence of wave events (“reflections”) = coherent space-time structures.

What features in the subsurface structure cause reflections to occur?

Abrupt (wavelength scale) changes in material mechanics act as internal boundaries, causing reflection of waves.

What is the mechanism through which this occurs?

Well logs: a “direct” view of the subsurface



Blocked logs from well in North Sea (thanks: Mobil R & D). Solid: p-wave velocity (m/s), dashed: s-wave velocity (m/s), dash-dot: density (kg/m^3). “Blocked” means “averaged” (over 30 m windows). Original sample rate of log tool < 1 m. **Reflectors** = jumps in velocities, density, **velocity trends**.

The Modeling Task

A useful model of the reflection seismology experiment must

- predict wave motion
- produce reflections from reflectors
- accomodate significant variation of wave velocity, material density,...

A *really good* model will also accomodate

- multiple wave modes, speeds
- material anisotropy
- attenuation, frequency dispersion of waves
- complex source, receiver characteristics

The Acoustic Model

Not *really good*, but good enough for this week and basis of most contemporary processing.

Relates $\rho(\mathbf{x})$= material density, $\lambda(\mathbf{x})$ = bulk modulus, $p(\mathbf{x},t)$= pressure, $\mathbf{v}(\mathbf{x},t)$ = particle velocity, $\mathbf{f}(\mathbf{x},t)$= force density (sound source):

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{f},$$

$$\frac{\partial p}{\partial t} = -\lambda \nabla \cdot \mathbf{v} \;\; (+\mathrm{i.c.'s, b.c.'s})$$

(compressional) wave speed $c = \sqrt{\frac{\lambda}{\rho}}$

acoustic field potential

$u(\mathbf{x},t) = \int_{-\infty}^{t} ds\, p(\mathbf{x},s)$:

$$p = \frac{\partial u}{\partial t}, \;\; \mathbf{v} = \frac{1}{\rho}\nabla u$$

Equivalent form: second order wave equation for potential

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = \int_{-\infty}^{t} dt\, \nabla \cdot \left(\frac{\mathbf{f}}{\rho}\right) \equiv \frac{f}{\rho}$$

plus initial, boundary conditions.

Theory

Weak solution of Dirichlet problem in $\Omega \subset \mathbf{R}^3$ (similar treatment for other b. c.'s):

$$u \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H^1_0(\Omega))$$

satisfying for any $\phi \in C_0^\infty((0,T) \times \Omega)$,

$$\int_0^T \int_\Omega dt\, dx \left\{ \frac{1}{\rho c^2} \frac{\partial u}{\partial t} \frac{\partial \phi}{\partial t} - \frac{1}{\rho} \nabla u \cdot \nabla \phi + \frac{1}{\rho} f \phi \right\} = 0$$

Theorem (Lions, 1972) Suppose that $\log \rho, \log c \in L^\infty(\Omega)$, $f \in L^2(\Omega \times \mathbf{R})$. Then weak solutions of Dirichlet problem exist; initial data

$$u(\cdot, 0) \in H^1_0(\Omega),\ \frac{\partial u}{\partial t}(\cdot, 0) \in L^2(\Omega)$$

uniquely determine them.

Further idealizations

- density is constant,
- source force density is *isotropic point radiator with known time dependence* ("source pulse" $w(t)$)

$$f(\mathbf{x}, t; \mathbf{x}_s) = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

$\Rightarrow$ acoustic potential, pressure depends on $\mathbf{x}_s$ also.

Forward map S = time history of pressure for each $\mathbf{x}_s$ at receiver locations $\mathbf{x}_r$ (predicted seismic data), depends on velocity field $c(\mathbf{x})$:

$$\mathcal{F}[c] = \{p(\mathbf{x}_r, t; \mathbf{x}_s)\}$$

Reflection seismic inverse problem

given *observed seismic data* d, find c so that

$$\mathcal{F}[c] \simeq d$$

This inverse problem is

- large scale - up to Tbytes, Pflops
- nonlinear
- yields to no known direct attack

Partial linearization

Almost all useful technology to date relies on partial linearization: write $c = v(1+r)$ and treat r as relative first order perturbation about v, resulting in perturbation of presure field $\delta p = \frac{\partial \delta u}{\partial t} = 0, t \leq 0$, where

$$\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta u = \frac{2r}{v^2} \frac{\partial^2 u}{\partial t^2}$$

Define **linearized forward map** F by

$$F[v]r = \{\delta p(\mathbf{x}_r, t; \mathbf{x}_s)\}$$

Analysis of $F[v]$ is the main content of contemporary reflection seismic theory.

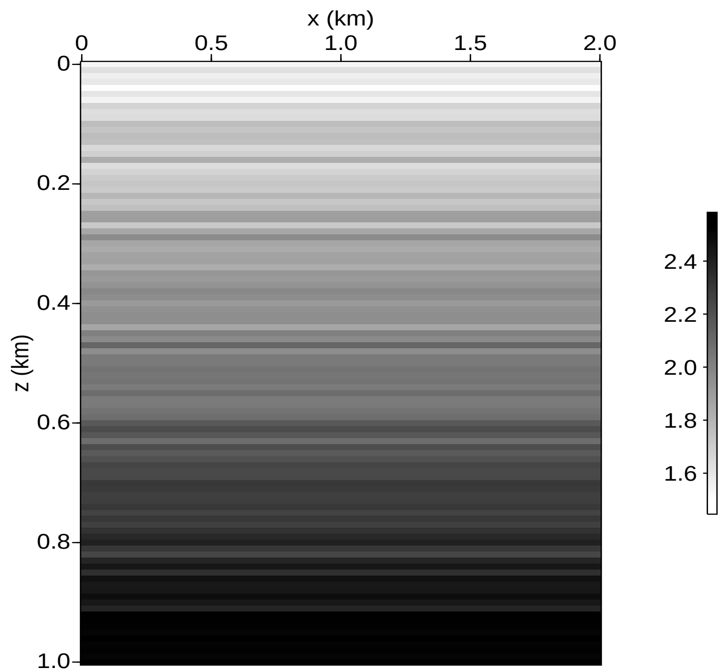
Linearization error

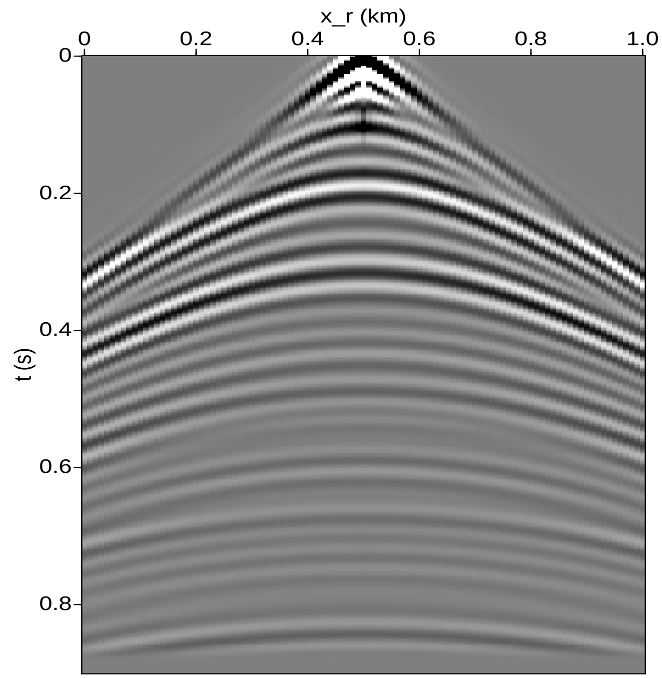
Critical question: If there is any justice $F[v]r$ = directional derivative $D\mathcal{F}[v][vr]$ of $\mathcal{F}$ - but in what sense? Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

$$\mathcal{F}[v(1+r)] - (\mathcal{F}[v] + F[v]r)$$

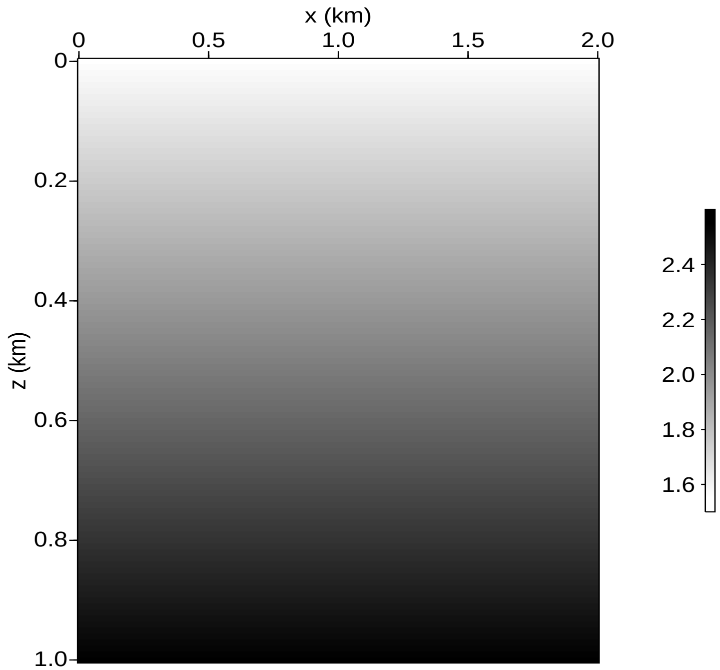
- *small* when v smooth, r rough or oscillatory on wavelength scale - well-separated scales
- *large* when v not smooth and/or r not oscillatory - poorly separated scales

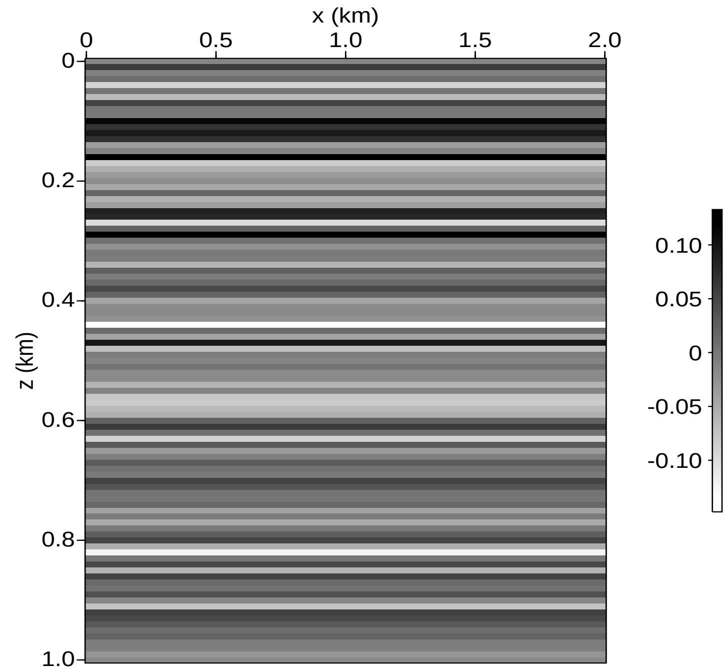
2D finite difference simulation: shot gathers with typical marine seismic geometry. Smooth (linear) $v(x,z)$, oscillatory (random) $r(x,z)$ depending only on z("layered medium"). Source wavelet $w(t)$ = bandpass filter.





Left: Total velocity $c = v(1 + r)$ with smooth (linear) background $v(x, z)$, oscillatory (random) $r(x, z)$. Std dev of r = 5%.
Right: Simulated seismic response ($\mathcal{F}[v(1 + r)]$), wavelet = bandpass filter 4-10-30-45 Hz. Simulator is (2,4) finite difference scheme.





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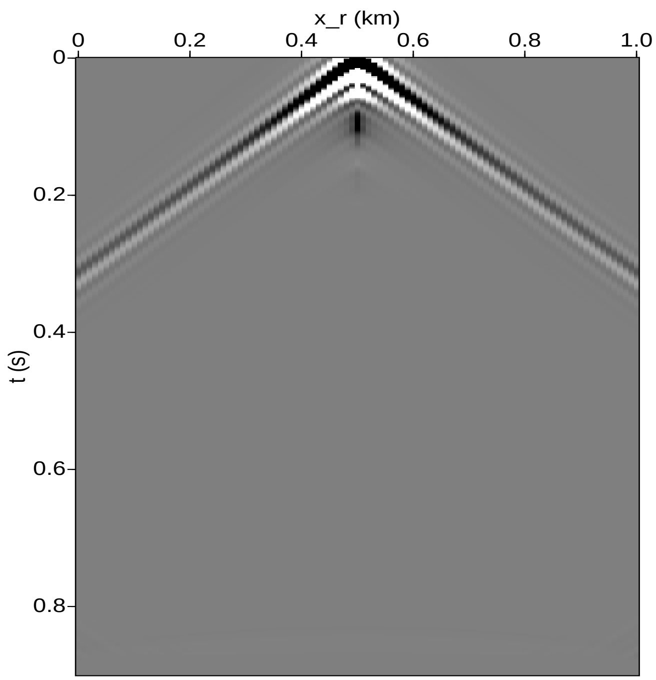
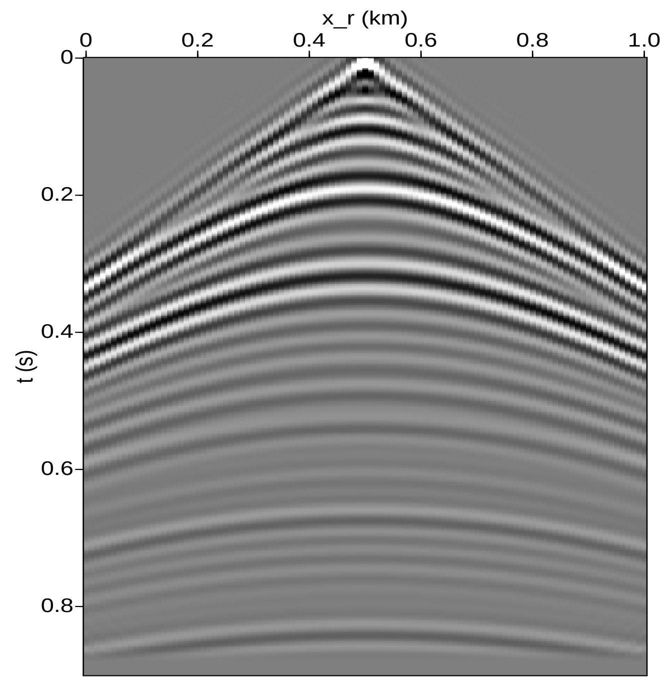
Model in previous slide as smooth background (left, $v(x, z)$) plus rough perturbation (right, $r(x, z)$).





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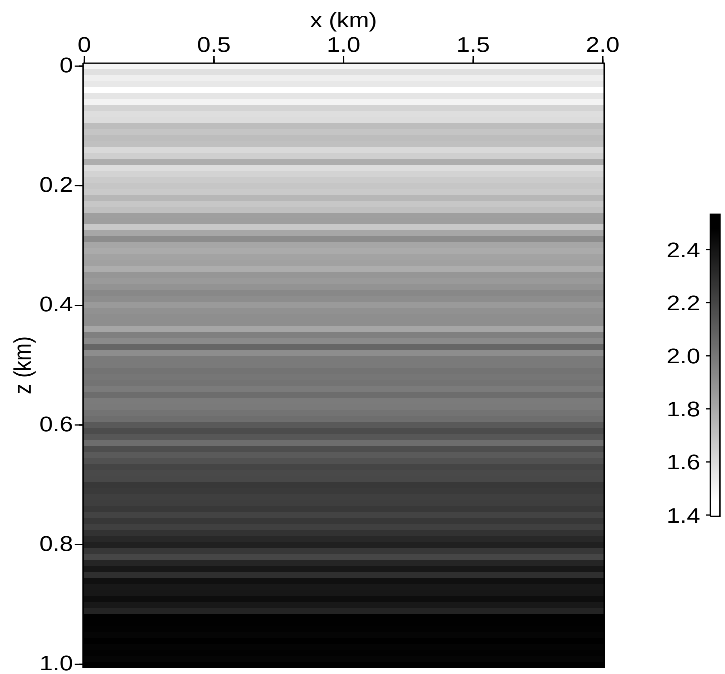
Left: Simulated seismic response of smooth model ($\mathcal{F}[v]$),
Right: Simulated linearized response, rough perturbation of smooth model ($F[v]r$)



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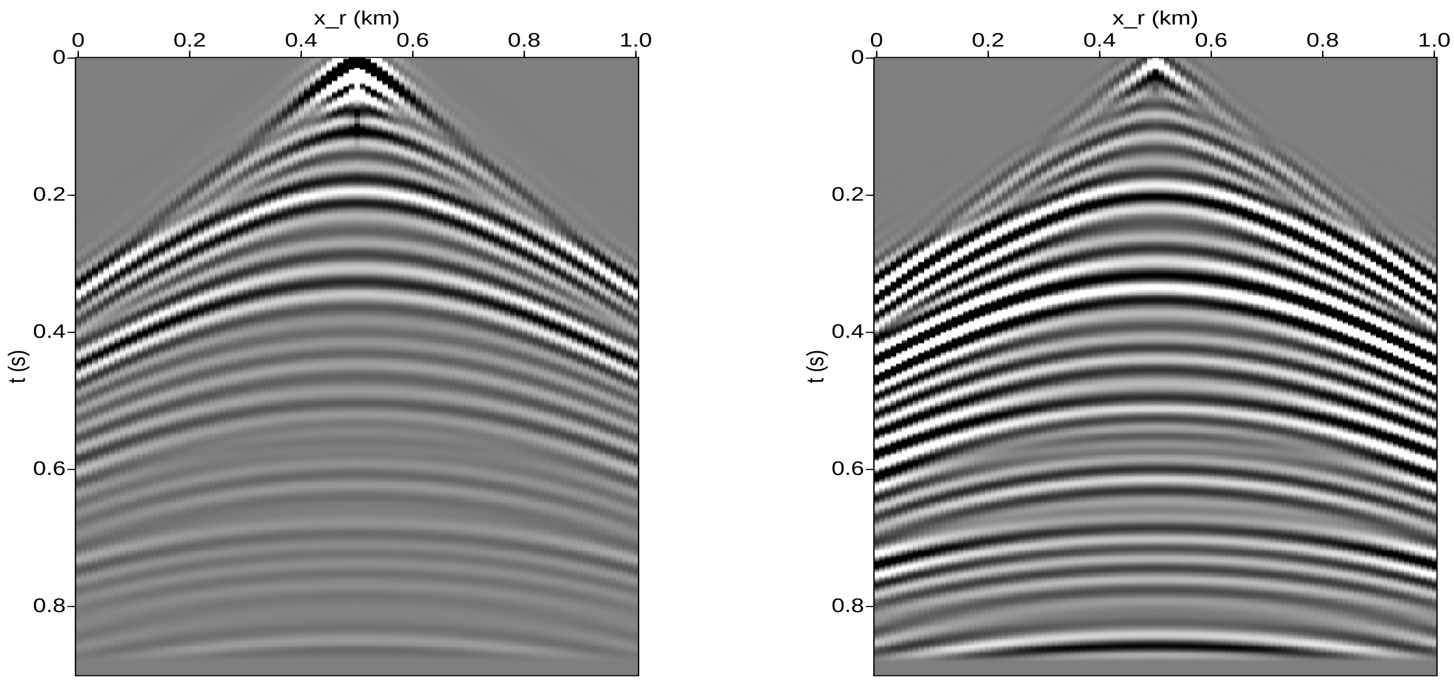
Model in previous slide as rough background (left, $v(x, z)$) plus smooth 5% perturbation ($r(x, z)$).



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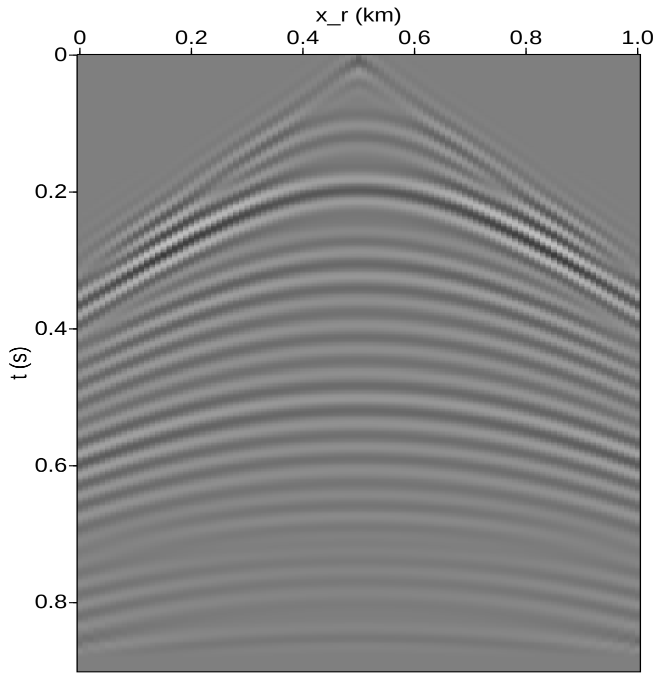
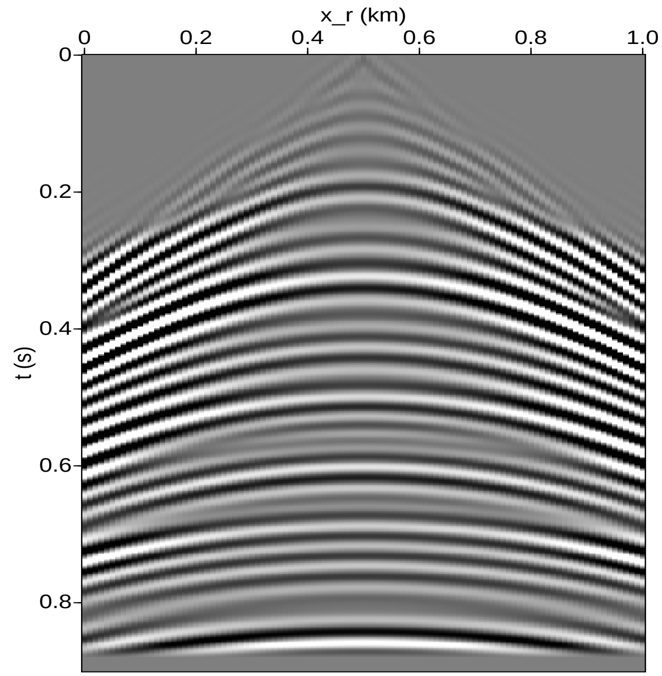
Left: Simulated seismic response of rough model ($\mathcal{F}[v]$),
Right: Simulated linearized response, smooth perturbation of rough model ($F[v]r$)





Left: linearization error ($\mathcal{F}[v(1+r)] - \mathcal{F}[v] - F[v]r$), rough perturbation of smooth background

Right: linearization error, smooth perturbation of rough background (plotted with same grey scale).

Summary

- v smooth, r oscillatory $\Rightarrow$ $F[v]r$ approximates **primary reflection** = result of wave interacting with material heterogeneity only once (single scattering); error consists of **multiple reflections**, which are “not too large” if r is “not too big”, and sometimes can be suppressed.
- v nonsmooth, r smooth $\Rightarrow$ error consists of *time shifts* in waves which are very large perturbations as waves are oscillatory.

No mathematical results are known which justify/explain these observations in any rigorous way, except in 1D.

Velocity Analysis and Imaging

Velocity analysis problem = partially linearized inverse problem: given d find v, r so that

$$S[v] + F[v]r \simeq d$$

Imaging problem = linear subproblem: given d and v, find r so that

$$F[v]r \simeq d - S[v]$$

Last 20 years:

- much progress on imaging
- much less on velocity analysis