FIFTH PROBLEM SESSION EXERCISES

(1) In general there is no unique solution for the inverse problem for the wave equation in one dimension and the purpose of this exercise is to explore why this is true. For a given source f with compact support contained in $t > 0$ and receiver position x_r the "forward problem" maps the wave velocity c and density ρ to the solution u of

$$\frac{1}{c^2(x)\rho(x)}\frac{\partial^2 u}{\partial t}(t,x) - \frac{\partial}{\partial x}\left(\frac{1}{\rho(x)}\frac{\partial u}{\partial x}(t,x)\right) = \frac{f}{\rho}, \quad u(t,x) = 0 \quad \text{for} \quad t < 0,$$

evaluated at x_r. Thus we have the map

$$\mathcal{F}[c,\rho](t) = u(t,x_r).$$

The inverse problem is to recover c and ρ from $\mathcal{F}[c,\rho]$.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism which is equal to the identity in a neighborhood of the support of f, and $\Phi(x_r) = x_r$. Show that

$$\mathcal{F}[(c \circ \Phi)\partial_x \Phi^{-1}, (\rho \circ \Phi)\partial_x \Phi] = \mathcal{F}[c,\rho].$$

in which $\circ$ denotes composition, i.e. $c \circ \Phi(x) = c(\Phi(x))$.

(2) For this exercise we refer to points in $\mathbb{R}^n$ using the notation $(x_1, \ldots, x_n)$. Calculate the wavefront set of the distribution defined on $\mathbb{R}^n$ by the function $H(x_1)$ where H is the Heaviside function. This completes the proof from lectures2.pdf of the fact that the wave front set of a function which jumps across a smooth hypersurface is the normal bundle of the hypersurface.

(3) Complete the proof of point (iv) from the lecture by showing (using the notation from lectures2.pdf) that for $\xi \in \mathrm{supp}(1-a)$, $\mathrm{supp}(\chi_1)$ small enough, and $x \in \mathrm{supp}(\chi_1)$,

$$|\tau \nabla \varphi(x) + \xi| > C\tau$$

for some constant $C > 0$ and all $\tau > 0$.

(4) Consider the distribution defined on $\mathbb{R}^2$ by the function

$$g(x_1,x_2) = \begin{cases} 1 & \text{if} \quad 0 < x_1 < 1, \quad 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Calculate the wave front set of this distribution.