INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS

Inverse Problems **19** (2003) 1113–1138 PII: S0266-5611(03)62524-4

True amplitude wave equation migration arising from true amplitude one-way wave equations

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Received 24 April 2003
Published 5 September 2003
Online at stacks.iop.org/IP/19/1113

Abstract

One-way wave operators are powerful tools for use in forward modelling and inversion. Their implementation, however, involves introduction of the square root of an operator as a pseudo-differential operator. Furthermore, a simple factoring of the wave operator produces one-way wave equations that yield the same travel times as the full wave equation, but do not yield accurate amplitudes except for homogeneous media and for almost all points in heterogeneous media. Here, we present augmented one-way wave equations. We show that these equations yield solutions for which the leading order asymptotic amplitude as well as the travel time satisfy the same differential equations as the corresponding functions for the full wave equation. Exact representations of the square-root operator appearing in these differential equations are elusive, except in cases in which the heterogeneity of the medium is independent of the transverse spatial variables. Here, we address the fully heterogeneous case. Singling out depth as the preferred direction of propagation, we introduce a representation of the square-root operator as an integral in which a rational function of the transverse Laplacian appears in the integrand. This allows us to carry out explicit asymptotic analysis of the resulting one-way wave equations. To do this, we introduce an auxiliary function that satisfies a lower dimensional wave equation in transverse spatial variables only. We prove that ray theory for these one-way wave equations leads to one-way eikonal equations and the correct leading order transport equation for the full wave equation. We then introduce appropriate boundary conditions at $z = 0$ to generate waves at depth whose quotient leads to a reflector map and an estimate of the ray theoretical reflection coefficient on the reflector. Thus, these true amplitude one-way wave equations lead to a 'true amplitude wave equation migration' (WEM) method. In fact, we prove that applying the WEM imaging condition to these newly defined wavefields in heterogeneous media leads to the Kirchhoff inversion formula for common-shot data when the one-way wavefields are replaced by their ray theoretic approximations. This extension enhances the original WEM method. The objective of that technique was a reflector map, only. The underlying theory did not address amplitude issues. Computer output obtained using numerically generated data confirms the accuracy of this inversion method. However, there are practical limitations. The observed data must be a solution of the wave equation. Therefore, the data over the entire survey area must be collected from a single common-shot experiment. Multi-experiment data, such as common-offset data, cannot be used with this method as currently formulated. Research on extending the method is ongoing at this time.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

One-way wave equations provide fast tools for modelling and migration. These one-way equations allow us to separate solutions of the wave equation into downgoing and upgoing waves except in the limit of near-horizontal propagation. The original one-way wave equations used for wave equation migration (WEM) (Claerbout 1971, 1985) were designed to produce accurate travel times, but were never intended to produce accurate amplitudes, even at the level of leading order asymptotic WKBJ or ray theoretic amplitudes. As such, that WEM provides a reflector map consistent with the background propagation model, but with unreliable amplitude information.

The objective of this paper is to describe a modification of those one-way wave equations to produce equations that provide an accurate leading order WKBJ or ray theoretic amplitude as well as an accurate travel time. The necessary modification of the basic one-way wave equations can be motivated by considering depth dependent ($v(z)$) media. In this case, through the use of Fourier transformation in time and transverse spatial coordinates (x, y), we reduce the problem of modifying the one-way equations to the study of ordinary differential equations. There, it is relatively simple to see how to modify the equations used by Claerbout in order to obtain equations that provide leading order WKBJ amplitudes, as well. This leading order amplitude is what we mean by 'true amplitude' for forward modelling.

For heterogeneous media, $v = v(x, y, z)$, the same one-way wave equations still provide true amplitudes. However, now the transverse wavevector (k_x, k_y) must be interpreted as differentiations in the corresponding dual spatial variables. Further, our modified one-way wave equations involve square roots and divisions by functions of this transverse wavevector. We provide an interpretation of these operators through some basic ideas from the theory of pseudo-differential operators.

We also provide a relatively simple representation of the one-way differential operators. This, in turn, allows us to prove that the ray theoretic solutions of these equations satisfy the separate eikonal equations for downgoing (increasing z) and upgoing waves, but the leading order amplitudes also satisfy the same equation—the *transport* equation—as does the leading order amplitude for the full wave equation. It is in this sense that we describe the solutions of these one-way wave equations as 'true amplitude' solutions.

Having these true amplitude one-way equations allows us to develop a 'true amplitude' WEM for heterogeneous media. To date, we only have numerical checks on this method for $v(z)$ media, where the pseudo-differential operators revert to simple multiplications in the temporal/transverse spatial Fourier domain. However, we are able to prove that the reflection amplitude agrees with the amplitudes generated by Kirchhoff inversion (true amplitude Kirchhoff migration) as developed by one of the authors (Bleistein 1987, Bleistein *et al* 2001) and colleagues. This proof is valid in heterogeneous media. Thus, at this time, the proof of validity is ahead of the computer implementation in terms of generality and it anticipates a reliable computer implementation in general heterogeneous media. It confirms that the output of this method is a reflector map with the peak amplitude on the reflector being in known proportion to an angularly dependent reflection coefficient at a specular reflection angle.

This type of inversion requires common-shot data with the receiver array covering the entire domain of the survey. This is a serious obstacle for practical implementation; such data gathers are still relatively rare. To date, we do not have an extension of this true amplitude WEM to other source/receiver configurations.

In the next section, we provide motivation for the modification of the simple one-way wave equations with the objective of developing true amplitude one-way equations for forward modelling. We start from homogeneous media where the basic one-way equations do provide the true amplitude. We then proceed to analysis of modelling for $v(z)$ media. That leads to a modification of the basic one-way equations in order to ensure that the appropriate transport equation is satisfied, as well.

Following that, we introduce the idea of using the same equations for heterogeneous media—$v = v(x, y, z)$—with functions of transverse wavevectors now reinterpreted as pseudo-differential operators. It is for this new interpretation that we provide a confirmation that these one-way wave equations provide true amplitude forward modelling. The proof of the claim that this extension leads to the appropriate transport equation is provided in appendix A. That extension and the proof were originally developed by the second author in Zhang (1993). Here, we present an update of that proof with attention to the application to WEM.

Following the discussion of forward modelling, we develop true amplitude WEM. This requires a modification of the basic one-way wave equations of Claerbout's WEM and also a modification of the boundary conditions of that WEM, which corrects the phase as well as the amplitude of the downgoing wave used in WEM. We also show how to modify Claerbout's original WEM equations in order to turn those into true amplitude equations. There is a subtlety of scaling by a pseudo-differential operator in the comparison. This leads to a slightly different one-way wave equation for the downgoing waves in Claerbout's approach when compared to the one-way equation that we use in the new theory.

We then provide a proof that this new approach leads to the same common-shot Kirchhoff inversion formula as is found in Bleistein (1987) and Bleistein *et al* (2001), as expressed by Keho and Beydoun (1988).

Following that, we present our numerical check of true amplitude WEM. As noted above, the examples that we present are for the case of a $v(z)$ medium where implementation of the pseudo-differential operators appearing in our method reduces to a multiplication in the temporal/transverse-spatial domain.

Except for the proofs provided in the appendices and in section 5, our exposition is mathematically informal and intuitive. It is our hope that this style will expand the readership beyond the usual limited mathematical community that is conversant with pseudo-differential operator theory.

2. Motivation

In this section we provide motivation for modifying the standard one-way wave equations that are used in WEM. We do this by starting with the standard operator factoring scheme for the wave equation in homogeneous media, separating off the z-dependence so that we can identify upgoing and downgoing waves. We identify the separate waves and confirm that they are solutions of first-order wave equations obtained by factoring the operator. We then show that the solutions of the derived one-way wave equations are no longer solutions of the full (two-way) wave equation when the medium is allowed to depend on z; that is, $v = v(z)$. In a WKBJ solution in a $v(z)$ medium, the amplitudes of the first-order equations do not agree with the amplitudes of the two one-way solutions of the full wave equation. By modifying the one-way wave equations, we obtain new equations whose eikonal and transport equations agree with the eikonal and transport equations for the full wave equation; each one-way equation governing one-way propagating waves of the two-way or full wave equation.

We begin by introducing the wave equation,

$$\frac{1}{v^2}\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = 0. \tag{1}$$

We will use the following definition of the Fourier transform:

$$F(x, y, z, t) = \frac{1}{(2\pi)^3}\int \mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}\omega\,\tilde{F}(k_x, k_y, z, \omega)\exp\{\mathrm{i}[\omega t - k_x x - k_y y]\}. \tag{2}$$

In this equation, the wavenumber integration ranges over all space. The frequency domain integral has the range $-\infty < \operatorname{Re}\omega < \infty$, with $\operatorname{Im}\omega$ large enough that the contour of integration passes below all singularities of the integrand in the complex ω-plane. Typically, this is just below the real axis in ω, with the integrand defined on the axis only through analytic continuation from below. Consequently, when viewed as an integral on the real ω-axis from $-\infty$ to ∞, we must interpret multi-valued functions, such as square roots, as if we had passed under their singular points—branch points and poles—to go from one side of a branch point to the other side.

Consistent with the Fourier transform in (2), when we write down WKBJ solutions, they will take the form

$$W = A\exp\{\mathrm{i}[\omega t - \Phi(x, y, z, \omega)]\}$$

or

$$W = A\exp\{\mathrm{i}\omega[t - \varphi(x, y, z)]\} \tag{3}$$

or

$$W = A\exp\{-\mathrm{i}\omega\varphi(x, y, z)\},$$

depending on the context. In these forms, $\nabla\Phi$ or $\nabla\varphi$ points in the direction of propagation of the wavefronts. In particular, $\operatorname{sgn}(\omega\partial\Phi/\partial z) = 1$ or $\operatorname{sgn}(\partial\varphi/\partial z) = 1$ indicates waves in the direction of increasing z; that is, *downgoing*. Of course, then, for upgoing waves $\operatorname{sgn}(\omega\partial\Phi/\partial z) = -1$ or $\operatorname{sgn}(\partial\varphi/\partial z) = -1$. We could alternatively represent upgoing waves by

$$W = A\exp\{\mathrm{i}[\omega t + \Phi(x, y, z, \omega)]\}$$

or

$$W = A\exp\{\mathrm{i}\omega[t + \varphi(x, y, z)]\}$$

or

$$W = A\exp\{\mathrm{i}\omega\varphi(x, y, z)\},$$

with $\operatorname{sgn}(\omega\partial\Phi/\partial z) = 1$ or $\operatorname{sgn}(\partial\varphi/\partial z) = 1$. Both alternatives for representing upgoing waves are used in this paper and in the literature.

For constant wave speed, we can rewrite this equation in the frequency/wavevector domain in the form

$$\mathcal{L}W = \frac{\partial^2 W}{\partial z^2} + k_z^2 W = \left[\frac{\partial}{\partial z} \mp \mathrm{i}k_z\right]\left[\frac{\partial}{\partial z} \pm \mathrm{i}k_z\right] W = 0. \tag{4}$$

Here,

$$k_z = \mathrm{sgn}(\omega)\sqrt{\frac{\omega^2}{v^2} - \bar{k}^2} = \frac{\omega}{v}\sqrt{1 - \frac{(v\bar{k})^2}{\omega^2}}, \tag{5}$$

and $\bar{k}$ is the transverse wavevector,

$$\bar{k} = (k_x, k_y), \qquad \bar{k}^2 = k_x^2 + k_y^2. \tag{6}$$

Further, the solutions of the full second-order wave equation are actually solutions of the two one-way wave equations

$$\left\{\frac{\partial}{\partial z} \pm \mathrm{i}k_z\right\} A_\pm \exp\{\mp \mathrm{i}k_z z\} = 0, \tag{7}$$

with the upper signs yielding a downgoing solution and the lower signs yielding an upgoing solution.

We would like one-way wave equations for the heterogeneous case, as well, similarly separating upward and downward propagating waves. For this generalization, we will content ourselves with ray theoretic solutions that yield the same leading order amplitude as does the two-way wave equation, using the leftmost expression in (4).

First consider the case where $v = v(z)$ only and recast the leftmost differential operator in (4) in a form that lends itself to ray theory analysis. To this end, we introduce the slowness vector $\bar{p}$ by setting

$$\bar{p} = \frac{\bar{k}}{\omega}, \qquad p_z = \frac{k_z}{\omega} = \frac{1}{v(z)}\sqrt{1 - (v(z)\bar{p})^2} \tag{8}$$

and rewrite the wave equation in (4) as

$$\frac{\partial^2 W}{\partial z^2} + \omega^2 p_z^2 W = 0. \tag{9}$$

In equations (8) and (9), if $(v\bar{k})^2$ exceeds ω^2, p_z is complex and the related solutions of (9) are termed *evanescent waves*. However, throughout this paper we are only interested in the propagation waves; that is, we focus on the solutions corresponding to $\omega^2 > (v\bar{k})^2$. When we substitute the standard form

$$W = A(z, \bar{p})\mathrm{e}^{-\mathrm{i}\omega\varphi(z,\bar{p})} \tag{10}$$

into (9), we find

$$\left\{-\omega^2\left[\left[\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right]^2 - p_z^2\right]A - \mathrm{i}\omega\left[2\frac{\mathrm{d}\varphi}{\mathrm{d}z}\frac{\mathrm{d}A}{\mathrm{d}z} + \frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2}A\right] + \mathrm{O}(1)\right\}\mathrm{e}^{-\mathrm{i}\omega\varphi} = 0. \tag{11}$$

Here, $\mathrm{O}(1)$ is in order of powers of $\mathrm{i}\omega$.

This last equation leads to the familiar eikonal and transport equations for the rays, except that we are in a Fourier transform domain in the transverse slowness vector, (p_x, p_y). Those equations are

$$\left[\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right]^2 = p_z^2 \Longrightarrow \frac{\mathrm{d}\varphi}{\mathrm{d}z} = \pm p_z, \qquad \text{and} \qquad 2\frac{\mathrm{d}\varphi}{\mathrm{d}z}\frac{\mathrm{d}A}{\mathrm{d}z} + \frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2}A = 0$$

or

$$\pm\left[2p_z\frac{\mathrm{d}A}{\mathrm{d}z} - \frac{1}{v^3(z)p_z}\frac{\mathrm{d}v(z)}{\mathrm{d}z}A\right] = 0 \tag{12}$$

or

$$\frac{\mathrm{d}A}{\mathrm{d}z} - \frac{1}{2v^3(z)p_z^2}\frac{\mathrm{d}v(z)}{\mathrm{d}z}A = 0.$$

Here, we think of the upper sign solution as the one for which $\mathrm{sgn}(p_z) = 1$. In this case, the upper sign corresponds to the downgoing wave and the lower sign corresponds to the upgoing wave. Note that the transport equation is the same for the two waves because only p_z^2 appears in that equation.

Now consider the two one-way wave equations in (7) and corresponding WKBJ or ray theoretic solutions. That is, consider

$$\left[\frac{\partial}{\partial z} \pm \mathrm{i}\omega p_z\right]A_\pm \mathrm{e}^{-\mathrm{i}\omega\varphi_\pm} = \mathrm{i}\omega\left[-\frac{\mathrm{d}\varphi_\pm}{\mathrm{d}z} \pm p_z\right]A_\pm \mathrm{e}^{-\mathrm{i}\omega\varphi_\pm} + \frac{\mathrm{d}A_\pm}{\mathrm{d}z}\mathrm{e}^{-\mathrm{i}\omega\varphi_\pm} = 0, \tag{13}$$

with consequent eikonal and transport equations

$$\frac{\mathrm{d}\varphi_\pm}{\mathrm{d}z} = \pm p_z \qquad \text{and} \qquad \frac{\mathrm{d}A_\pm}{\mathrm{d}z} = 0. \tag{14}$$

While the eikonal equations in (12) and (14) agree, the transport equations for amplitudes do not. Thus, we must consider modifying the one-way wave equations in (13) if we are to make the latter transport equation agree with the former, while keeping the same eikonal equation in both. We can achieve this goal for the one-way wave equations if we modify them by adding a term to the one-way operators appearing in the leftmost expression in (13). The key to doing this comes from examining the last form of the transport equation in (12). That is, we consider the new one-way wave equations

$$\left\{\frac{\partial}{\partial z} \mp \mathrm{i}\omega p_z - \frac{1}{2v^3(z)p_z^2}\frac{\mathrm{d}v(z)}{\mathrm{d}z}\right\}W = 0. \tag{15}$$

For these equations

$$\begin{aligned}\left\{\frac{\partial}{\partial z} \pm \mathrm{i}\omega p_z - \frac{1}{2v^3(z)p_z^2}\frac{\mathrm{d}v(z)}{\mathrm{d}z}\right\}&A_\pm \mathrm{e}^{-\mathrm{i}\omega\varphi_\pm}\\ &= \mathrm{i}\omega\left[-\frac{\mathrm{d}\varphi_\pm}{\mathrm{d}z} \pm p_z\right]A_\pm \mathrm{e}^{-\mathrm{i}\omega\varphi_\pm} + \left[\frac{\mathrm{d}A_\pm}{\mathrm{d}z} - \frac{1}{2v^3(z)p_z^2}\frac{\mathrm{d}v(z)}{\mathrm{d}z}A_\pm\right]\mathrm{e}^{-\mathrm{i}\omega\varphi_\pm} = 0,\end{aligned}$$

for which the transport equation in (14) is replaced by

$$\frac{\mathrm{d}A_\pm}{\mathrm{d}z} - \frac{1}{2v^3(z)p_z^2}\frac{\mathrm{d}v(z)}{\mathrm{d}z}A_\pm = 0 \tag{16}$$

while the eikonal equation remains unchanged. We have already seen that this is equivalent to the transport equation for the full wave equation. Hence, the one-way wave operators in (15) will produce the upgoing and downgoing travel times and leading order amplitudes of the full wave equation. In fact, these solutions are exact for the case of a medium that depends only on z; that is, when $v = v(z)$.

The question now arises as to how this insight can be extended to factoring the original wave operator in (1) of a fully heterogeneous medium, where $v = v(x, y, z)$. In this case, the Fourier transverse transform technique used above will not provide an easy problem in the Fourier domain; the product in the first term leads to a convolution in the transverse Fourier domain. Thus, we could not have derived a differential equation in z alone through Fourier transformation. If we disregard this obstacle, we could still examine the one-way wave equations in (15) with the hope of achieving a sensible interpretation. To that end, let us first recast the one-way wave equations (15) in the original Fourier variables. Thus, we first rewrite that equation as

$$\left\{\frac{\partial}{\partial z} \pm \mathrm{i}k_z - \frac{\omega^2}{2v^3(x,y,z)k_z^2}\frac{\partial v(x,y,z)}{\partial z}\right\}W = 0. \tag{17}$$

For reasons that will become clear in the next section, we prefer writing the multiplier on this additional term as follows:

$$\frac{\omega^2}{2v^3(x,y,z)k_z^2}\frac{\partial v(x,y,z)}{\partial z} = \frac{1}{2v(x,y,z)}\frac{\partial v(x,y,z)}{\partial z}\left[1 + \frac{(v(x,y,z)\bar{k})^2}{\omega^2 - (v(x,y,z)\bar{k})^2}\right]$$

and then rewrite (17) as

$$\left\{\frac{\partial}{\partial z} \mp \mathrm{i}k_z - \frac{1}{2v(x,y,z)}\frac{\partial v(x,y,z)}{\partial z}\left[1 + \frac{(v(x,y,z)\bar{k})^2}{\omega^2 - (v(x,y,z)\bar{k})^2}\right]\right\}W = 0. \tag{18}$$

Let us now think of ω as a place-holder for the temporal derivative and $-\bar{k} = -(k_x, k_y)$ as a place-holder for a transverse gradient operator; that is,

$$\mathrm{i}\omega \quad \Leftrightarrow \quad \partial/\partial t \qquad \text{and} \qquad \mathrm{i}(k_x, k_y) \quad \Leftrightarrow \quad -(\partial/\partial x, \partial/\partial y).$$

Then we could easily give meaning to the expression $(v(x,y,z)\bar{k})^2$ as follows:

$$(v(x,y,z)\bar{k})^2 \quad \Leftrightarrow \quad -(v(\partial/\partial x, \partial/\partial y)) \cdot (v(\partial/\partial x, \partial/\partial y)).$$

However, symbolically, k_z involves taking the square root of a differential operator, while the division in the last term requires that we give meaning to the reciprocal of a differential operator. Interpretation of such expressions is what the theory of pseudo-differential operators is all about. Thus, in the next section we address the interpretation of these terms in fully heterogeneous media where $v = v(x,y,z)$. With an appropriate interpretation, it turns out that these modified one-way wave equations provide an asymptotic solution for fully heterogeneous media—$v = v(x,y,z)$—for which the resulting transport equations agree with the transport equation for the full two-way wave equation.

3. True amplitude one-way wave propagation

Motivated by the discussion of the previous section, we introduce the same one-way equations (18) for the heterogeneous medium in which $v = v(x,y,z)$. Here, we will extend the definition of differentiation through the use of pseudo-differential operators so that we can give meaning to the pseudo-differential operator k_z in (17) and thereby give meaning to those one-wave equations themselves.

Couched in the language of pseudo-differential operator theory, we rewrite those one-way wave equations (18) as

$$\mathcal{L}_\pm W = \left[\frac{\partial}{\partial z} \pm \Lambda\right]W - \Gamma W = 0. \tag{19}$$

Here, Λ and Γ are pseudo-differential operators with symbols λ and γ, respectively:

$$\begin{aligned}\lambda &= \mathrm{i}k_z = \frac{\mathrm{i}\omega}{v}\sqrt{1 - \frac{(v\bar{k})^2}{\omega^2}},\\ \gamma &= \frac{1}{2v}\frac{\partial v}{\partial z}\left(1 + \frac{(v\bar{k})^2}{\omega^2 - (v\bar{k})^2}\right) = \frac{v_z}{2v}\left(1 + \frac{(v\bar{k})^2}{\omega^2 - (v\bar{k})^2}\right),\end{aligned} \qquad \omega^2 > (v\bar{k})^2. \tag{20}$$

In this last expression, the subscript in the variable v_z denotes taking the derivative with respect to z, in contrast to its use in k_x, k_y, k_z where it distinguishes the three components of the wavevector.

In (20), the notation $(v\bar{k})^2$ is the symbolic representation for the operator

$$\begin{aligned}- (v\nabla_{Tx})^2 &= -(v(\vec{\rho}, z)\nabla_{Tx})^2 = -\left(v(\vec{\rho}, z)\frac{\partial}{\partial x}\right)^2 - \left(v(\vec{\rho}, z)\frac{\partial}{\partial y}\right)^2 \\ &= -v^2(\vec{\rho}, z)\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] - v(\vec{\rho}, z)\left[\frac{\partial v(\vec{\rho}, z)}{\partial x}\frac{\partial}{\partial x} + \frac{\partial v(\vec{\rho}, z)}{\partial y}\frac{\partial}{\partial y}\right],\end{aligned} \tag{21}$$

where $\vec{\rho} = (x, y)$. In the above definition of $(v\nabla_{Tx})^2$, we have included the lower order differential operator $v(v_x\partial_x + v_y\partial_y)$. We do this intentionally for the purpose of making the derived one-way wave equations (19) produce the same leading order asymptotic amplitude as the full wave equation (1) does. As we have shown in section 2 for the $v(z)$ case, in order to preserve the amplitude, we need operators in one-way wave equations to be valid *up to two orders in ω* in asymptotic expansions, rather than the usual leading order. For a general heterogeneous medium $v(x, y, z)$, since the exact operator decomposition is elusive, we must be very careful to identify those terms in the operators that contribute to the leading order amplitude and have to be retained. Other terms that only affect lower order amplitude corrections which are outside our area of interest can be safely discarded. These issues will arise in appendix A when we prove that the newly defined one-way wave equations actually preserve the leading order amplitude. Readers are strongly recommended to read appendix A in order to get the full picture of this work.

Returning to the discussion of Λ, its symbol λ has an *exact* representation (Zhang 1993) in terms of a rational function of the argument inside the square root, that is,

$$\lambda = \mathrm{i}k_z = \frac{\mathrm{i}\omega}{v}\left\{1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1 - s^2}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,\mathrm{d}s\right\}, \qquad \omega^2 > (v\bar{k})^2. \tag{22}$$

A proof of this identity using contour integration in the complex plane is provided in appendix B. Such a proof shows that the above identity is also valid for waves defined in $\omega^2 < (v\bar{k})^2$; this corresponds to evanescent waves, ignored in our discussion.

It is fairly easy to think of the symbol $\omega^2 - (v\bar{k})^2$ as representing the two-way wave operator. That is,

$$- [\omega^2 - s^2(v\bar{k})^2] \quad \Leftrightarrow \quad \mathrm{L}_T(s; \vec{\rho}, z, t) = \frac{\partial^2}{\partial t^2} - s^2(v\nabla_{Tx})^2, \tag{23}$$

in which case, this operator appearing in the denominator of the symbols would represent an *inverse differential operator* or convolution with a Green function for the adjoint of this operator.

Further, if we neglect the transverse dependence in Λ, neglect the amplitude corrections in Γ and revert to constant velocity for a moment, then the identity (22) used in the one-way wave equations in (19) leads to the equations

$$\left[\frac{\partial}{\partial z} \pm \frac{1}{v}\frac{\partial}{\partial t} + \cdots\right]W = 0,$$

with solutions

$$W = F(z \mp vt + \cdots).$$

That is, the choices of signs that we have made in the symbolic operators ensure the separation into downgoing and upgoing waves that we intended.

Symbolically then, we think of the operators Λ and Γ as follows:

$$\begin{aligned}\Lambda &= \frac{1}{v}\frac{\partial}{\partial t}\left\{ I - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\mathrm{L}_T^{-1}(s;\vec{\rho},z,t)(v\nabla_{Tx})^2\,\mathrm{d}s\right\},\\ \Gamma &= \frac{v_z}{2v}(I+\mathrm{L}_T^{-1}(1;\vec{\rho},z,t)(v\nabla_{Tx})^2).\end{aligned} \tag{24}$$

Now we can see that the representation (18) leads to the same inverse of the operator L_T, evaluated at general s or $s = 1$ in the two pseudo-differential operator expressions used in our one-way wave equations.

Let us introduce a function $q(s;\vec{\rho},z,t)$ that satisfies the equation

$$\begin{aligned}\mathrm{L}_T(s;\vec{\rho},z,t)q &= \left\{\frac{\partial^2}{\partial t^2} - s^2(v\nabla_{Tx})^2\right\}q(s;\vec{\rho},z,t)\\ &= (v\nabla_{Tx})^2 W(\vec{\rho},z;t) \qquad z>0, \quad t>0.\end{aligned} \tag{25}$$

Using this function plus the identity (22) for k_z, ΛW and ΓW can be expressed as

$$\Lambda W = \frac{1}{v}\frac{\partial W}{\partial t} - \frac{1}{\pi v}\frac{\partial}{\partial t}\int_{-1}^{1}\sqrt{1-s^2}q(s;\vec{\rho},z,t)\,\mathrm{d}s, \qquad \Gamma W = \frac{v_z}{2v}[W + q(1;\vec{\rho},z,t)]. \tag{26}$$

Here, the arguments of W are $(\vec{\rho},z,t)$. Furthermore, (19) can be rewritten in the expanded form

$$\mathcal{L}_{\pm}W = \frac{\partial W}{\partial z} \pm \frac{1}{v}\frac{\partial W}{\partial t} \mp \frac{1}{\pi v}\frac{\partial}{\partial t}\int_{-1}^{1}\sqrt{1-s^2}q(s;\vec{\rho},z,t)\,\mathrm{d}s + \frac{v_z}{2v}[W+q(1;\vec{\rho},z,t)] = 0. \tag{27}$$

We assume that boundary data

$$W(\vec{\rho},0,t) = W_0(\vec{\rho},t), \tag{28}$$

are given. Further, for the downgoing wave, we provide the initial condition $W(\vec{\rho},z,0) = 0$, while for the upcoming wave, we provide a *final* condition of zero. That is, $W(\vec{\rho},z,t) = 0$ for t greater than some finite time, $t > T$. For the inverse problem, we need both one-way operators. The source propagates downward, subject to the wave equation (19) with upper signs and a boundary value at $z = 0$ that is equivalent to an impulsive source. On the other hand, the observed data are governed by the upward propagating one-way wave equation, lower signs in (19) with the given data being the appropriate observed data at $z = 0$. That is the subject of the next section. Here, we have only introduced the governing equations for propagation.

It is for this expanded form of the one-way wave equations that the second author has shown in Zhang (1993) that the transport equations for the one-way wave equations agree with the transport equation for the full wave equation. Of course, this is in addition to the agreement of the eikonal equations, except for the separation into downgoing and upgoing provided explicitly by the one-way wave equations. A modified version of that proof is provided in appendix A.

The above derivation of one-way wave equations is rather symbolic and intuitive. Some issues regarding rigour have been addressed in Zhang (1993), in which the derivation was based on wave operator decomposition by symbol calculus. To make the symbol calculus in (20) valid, the velocity function v should belong to the symbol class S^0 which is defined in Treves (1980). However, for the existence of the solutions W and q in equations (27) and (25), we can ease this restriction and only require v to be piecewise smooth.

4. True amplitude wave equation migration

In this section, we describe the application of these true amplitude one-way wave equations to WEM. The objective is to derive a true amplitude WEM. We begin by introducing Claerbout's (1971, 1985) classic WEM and explain how we modify the governing equations and boundary data to obtain our proposed true amplitude WEM.

The standard method uses the one-way propagators of (4), even for heterogeneous media. More specifically, suppose that the reflected wavefield from a single source experiment is observed at $z = 0$ for all time. Then the source and observed wavefields are assumed to be solutions of the equations

$$\begin{aligned} &\left(\frac{\partial}{\partial z} + \Lambda\right) D = 0, \\ &D(x, y, z = 0; \omega) = -\delta(\vec{x} - \vec{x}_s), \qquad \vec{x} = (x, y, z), \quad \vec{x}_s = (x_s, y_s, 0), \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\left(\frac{\partial}{\partial z} - \Lambda\right) U = 0, \\ &U(\vec{x}_s; \omega) = Q(x, y; \omega), \end{aligned} \tag{30}$$

where D is the downgoing (source) wavefield and U is the upgoing (observed) wavefield. The image is then produced as an *impedance* or *reflectivity* function at every image point defined by

$$R(x, y, z) = \frac{1}{2\pi} \int \frac{U(\vec{x}; \omega)}{D(\vec{x}; \omega)} \, \mathrm{d}\omega. \tag{31}$$

The key to this imaging method is that the constructive/destructive interference between the phases of the two waves produces a large amplitude where the reflectors reside and a small amplitude where they do not. While this result produces a reflector map, it does not provide accurate amplitude information. To achieve that, we use the solutions of our modified true amplitude one-way wave equations (19). That is, we introduce p_D and p_U as solutions of the following problems:

$$\begin{aligned} &\left(\frac{\partial}{\partial z} + \Lambda - \Gamma\right) p_D(\vec{x}; \omega) = 0, \\ &p_D(\vec{x}_s; \omega) = -\tfrac{1}{2}\Lambda^{-1}\delta(\vec{x} - \vec{x}_s), \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\left(\frac{\partial}{\partial z} - \Lambda - \Gamma\right) p_U(\vec{x}; \omega) = 0, \\ &p_U(\vec{x}_s; \omega) = Q(x, y; \omega). \end{aligned} \tag{33}$$

Here, we have not only modified the governing one-way wave equations in accordance with the discussion of the previous sections, but we have also modified the boundary term for the downgoing wave from the source. The reason for this is that the boundary value only accounts for the impulsive nature of the source in the transverse direction. In the z-direction, we must account for an impulsive source by balancing the terms

$$\frac{\partial^2 u}{\partial z^2} \qquad \text{and} \qquad -\delta(z)$$

or

$$\frac{\partial u}{\partial z} \qquad \text{and} \qquad -H(z)$$

or

$$\Lambda u \qquad \text{and} \qquad 1.$$

(In the second balance, $H(z)$ is the Heaviside function.) We think of the impulsive source as sending half its energy in each direction in z. Hence, in the positive z-direction we use half of the balance in this last expression as the boundary value for the downgoing wave (33). Note that this modification introduces a phase shift in this wave since Λ is imaginary and carries the same sign as ω. See also Wapenaar (1990) for a more rigorous exposition of this point.

Also, we modify the imaging condition (31) to be the quotient of the wavefields p_D and p_U:

$$R(\vec{x}) = \frac{1}{2\pi}\int \frac{p_U(\vec{x};\omega)}{p_D(\vec{x};\omega)}\,\mathrm{d}\omega. \tag{34}$$

See Zhang *et al* (2001, 2002).

The introduction of the operator Λ^{-1} in (32) requires that we give meaning to this pseudo-differential operator, just as we did for Λ itself. We do this in appendix B.

As an alternative to dealing with Λ^{-1} in the boundary condition, it might be easier to solve a modified version of (29) for D, after adjusting that equation to be equivalent to the true amplitude equation (32) for p_D. Let us therefore introduce a new D in (32):

$$D = 2\Lambda p_D, \qquad \text{or} \qquad p_D = \tfrac{1}{2}\Lambda^{-1}D, \tag{35}$$

for which the boundary data agree with the data for D in (29). Now, let us substitute this choice into the differential equation (32) for p_D:

$$\left(\frac{\partial}{\partial z} + \Lambda - \Gamma\right)\Lambda^{-1}D = \Lambda^{-1}\frac{\partial D}{\partial z} - \Lambda^{-2}\frac{\partial \Lambda}{\partial z}D + D - \Gamma\Lambda^{-1}D = 0,$$

$$\Rightarrow \frac{\partial D}{\partial z} + \Lambda D - \left[\Lambda^{-1}\frac{\partial \Lambda}{\partial z} + \Lambda\Gamma\Lambda^{-1}\right]D = 0. \tag{36}$$

We have omitted the constant factor of 2 throughout these manipulations. For the last two terms, the leading order contributions contribute to the leading order amplitude and the corrections affect only lower order amplitudes. Therefore we can set

$$\Lambda\Gamma\Lambda^{-1} \approx \Gamma$$

$$\Lambda^{-1}\frac{\partial \Lambda}{\partial z} + \Gamma \quad \Leftrightarrow \quad \frac{1}{k_z}\frac{\partial k_z}{\partial z} - \frac{1}{2k_z}\frac{\partial k_z}{\partial z} = -\gamma.$$

Consequently, the true amplitude equation for D is

$$\left(\frac{\partial}{\partial z} + \Lambda + \Gamma\right)D = 0. \tag{37}$$

Within the factor of 2, this use of D agrees with the earlier papers cited in the references. Thus, we can avoid applying a pseudo-differential operator to the two-dimensional Dirac delta function in the boundary condition for p_D in (32) by solving for D, instead, using this equation and the boundary condition in (29). Of course, we could tie U and p_U together in exactly the same way. That is,

$$U = 2\Lambda p_U,$$

leading to the differential equation

$$\left(\frac{\partial}{\partial z} - \Lambda + \Gamma\right)U = 0.$$

However, now the pseudo-differential operator would appear in the boundary data for U; that is,

$$U(\vec{x}_s;\omega) = 2\Lambda Q(x, y;\omega).$$

Furthermore, the correct imaging condition now becomes

$$R(\vec{x}) = \frac{1}{2\pi}\int \frac{p_U}{p_D}\,\mathrm{d}\omega = \frac{1}{2\pi}\int \frac{\Lambda^{-1}U}{\Lambda^{-1}D}\,\mathrm{d}\omega.$$

5. Comparison of true amplitude WEM output and Kirchhoff inversion output

The previous section proposed the use of new one-way propagators for the surface data, a modification of the data for the downgoing wave and a new imaging condition in equation (34) to achieve true amplitude WEM. Here we show that the integration in this imaging condition agrees with the integration for Kirchhoff inversion in Bleistein (1987) and Bleistein *et al* (2001). Actually, we derive the representation of the Kirchhoff inversion formula as stated in Keho and Beydoun (1988). In carrying out this comparison, we rely on the proof of appendix A of the dynamic as well as the kinematic equivalence of the solutions of the one-way wave equations and the solutions of the two-way wave equation.

We start by considering the problem for p_D as defined by (32). Relying on the proof of appendix A, we know that this function is dynamically equivalent to the downgoing (outward radiating) Green function of the full wave equation. Therefore,

$$p_D(\vec{x}, \vec{x}_s; \omega) = A(\vec{x}, \vec{x}_s)\mathrm{e}^{-\mathrm{i}\omega\varphi(\vec{x},\vec{x}_s)}. \tag{38}$$

Here, φ is the solution of the eikonal equation for the full wave equation (1) with $\partial\varphi/\partial z > 0$ and A is the solution of the transport equation for the full wave equation. Equivalently, φ is a solution of the eikonal equation

$$\frac{\mathrm{d}\varphi_\pm}{\mathrm{d}z} = \sqrt{\frac{1}{v^2(\vec{x})} - p_x^2 - p_y^2},$$

deduced from (18) with the upper sign, and A is a solution of the transport equation

$$2\nabla\varphi_\pm \cdot \nabla A_\pm + A_\pm \Delta\varphi_\pm = 0,$$

deduced from (18) with the upper sign. This is the essential conclusion of the proof of appendix A.

To derive a representation of the function p_U, we have to work a little harder. Again, however, we rely on the equality between the leading order asymptotic solutions of the one-way wave equation, (33), and the full wave equation. In appendix C, we show that the Green function representation for p_U is given by

$$p_U(\vec{x}; \omega) = 2\mathrm{i}\omega \int \frac{\cos\alpha_r}{v(\vec{x}_r)} A(\vec{x}_r, \vec{x})\mathrm{e}^{\mathrm{i}\omega\varphi(\vec{x}_r,\vec{x})}\,\mathrm{d}x_r\,\mathrm{d}y_r, \qquad \vec{x}_r = (x_r, y_r, 0). \tag{39}$$

In this equation, again, φ and A are the phase and amplitude of the free space Green function for the full wave equation. However, because p_U is an upward (incoming) wave, we need the inward propagating Green function. Hence the sign in the phase is opposite the sign of the phase of p_D defined by (38). Further, $v_r = v(\vec{x}_r)$ and α_r is the emergence angle of the ray with respect to the normal.

We use this last result and (38) in equation (34) for R and obtain

$$R(\vec{x}) = 2\int\int \mathrm{i}\omega \frac{\cos\alpha_r}{v_r} \frac{A(\vec{x}_r, \vec{x})}{A(\vec{x}, \vec{x}_s)} \mathrm{e}^{\mathrm{i}\omega\{\varphi(\vec{x}_r,\vec{x})+\varphi(\vec{x},\vec{x}_s)\}}\,\mathrm{d}x_r\,\mathrm{d}y_r\,\mathrm{d}\omega. \tag{40}$$

This is the result in Bleistein (1987) and Bleistein *et al* (2001) as expressed by Keho and Beydoun (1988).

This representation of the imaging condition for true amplitude WEM, being the same as the inversion formula for Kirchhoff inversion, confirms the claim that this formulation provides a true amplitude WEM. We can now be assured that the output of this new WEM will provide a reflectivity map with peak amplitude on the reflector being a known multiple of the geometrical optics reflection coefficient. The incidence angle in that reflection coefficient is the angle defined by the specular pair of rays from a source/receiver pair. Note that this

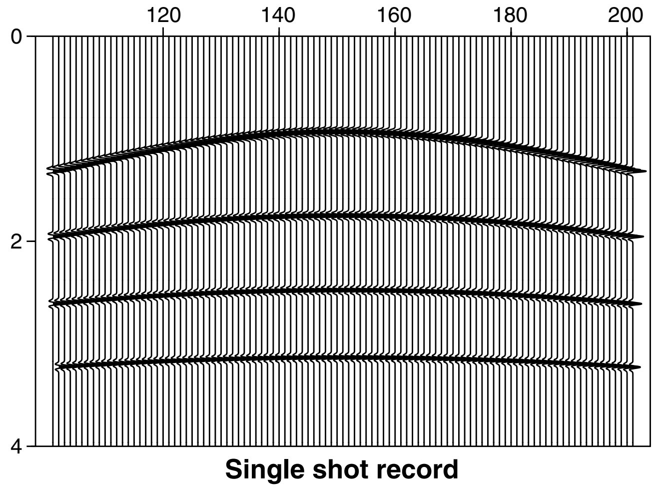

Figure 1. Input model data.

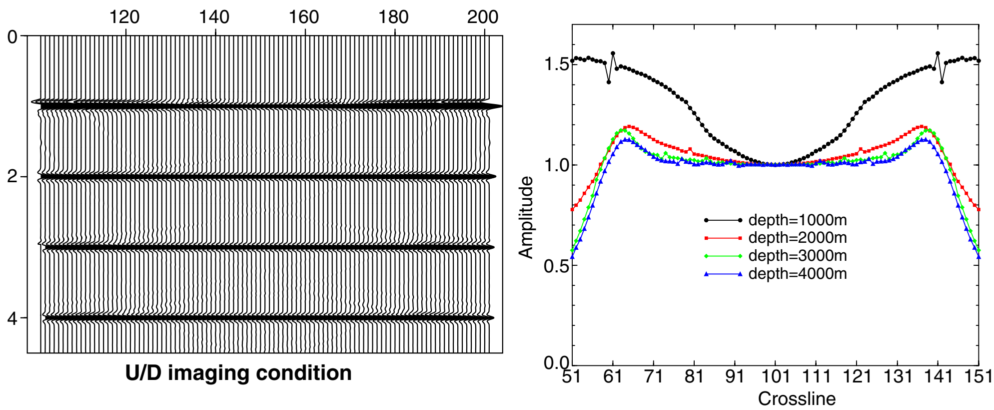

Figure 2. Left: finite difference migration obtained using (31) for imaging. Right: peak amplitudes along the four reflectors. The wide angle error decreases with the depth of the reflector.

result is confirmed for heterogeneous media—$v = v(\vec{x})$. Previous verifications, as presented for example in Zhang *et al* (2001, 2002), were only for the case of depth dependent media—$v = v(z)$.

6. Numerical tests

To show how true amplitude common-shot migration works, we apply it to a 2D horizontal reflector model in a medium with velocity $v = 2000 + 0.3z$. Recall from the theory that in this case, the modelling and migration can be carried out in the transverse spatial and temporal Fourier domains, with $\bar{k}$ being the simple transverse part of the wavevector.

The input data (figure 1) is a single shot record over four horizontal reflectors from unit density contrast.

Figure 2, left, shows the migrated shot record obtained using the conventional common-shot migration algorithm (31). The peak amplitudes along the four migrated reflectors are

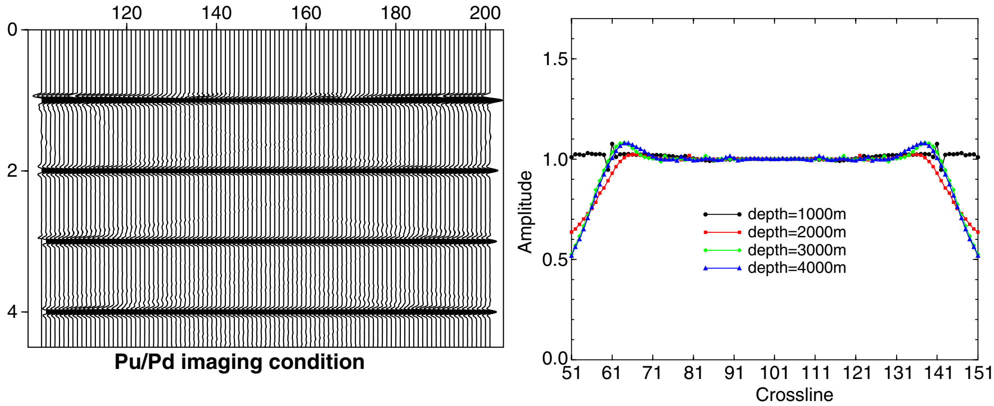

Figure 3. Left: finite difference migration obtained using (34) for imaging. Right: peak amplitudes along the four reflectors. The wide angle error decreases with the depth of the reflector.

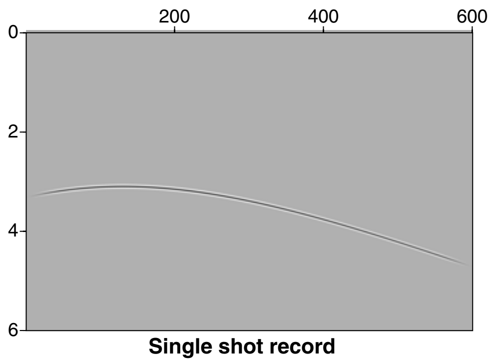

Figure 4. The shot record for the common-shot response from a dipping plane.

shown in figure 2 (right). This method has a phase error: note the multiplication by i in Λ on the right side in (32) as opposed to the lack of such a phase shifting factor on the right side of (29). The consequent phase error has been corrected during the migration. However, the migrated amplitudes are poor, especially on the reflector at depth $z = 1000$ m along which the ray directions vary over a wide range. The wide angle peak amplitudes decrease monotonically with increasing depth. The greatest error occurs at wide angle, with the result along the shallowest reflector being the worst. However, the error is zero at zero offset; in this limit, $\bar{k} = (0, 0)$ and $\cos\alpha_r = 1$ in (40).

The scaling in the boundary condition in (32) provides directivity to the downgoing wave in the solution of (32) that is lacking in the solution to (29).

Figure 3 (left) shows results of true amplitude common-shot migration (34). The peak amplitudes along the reflectors are shown in figure 3 (right).

Using (40) as a guide, we see that the true amplitude output is proportional to the ratio of amplitudes of the Green functions from source and receiver to the output point. Because of the symmetry of this problem, the Green function amplitudes from the source and the receiver are equal, with quotient equal to unity at the stationary point of the integral that produces the output. In fact, the ratio of amplitudes obtained using the Claerbout migration equations is also equal to one. Thus, the error in the Claerbout output depicted in figure 2 does not depend on the difference between the amplitudes produced by the two methods and can only depend on the difference between the scalings of the source wave in equations (29) and (32).

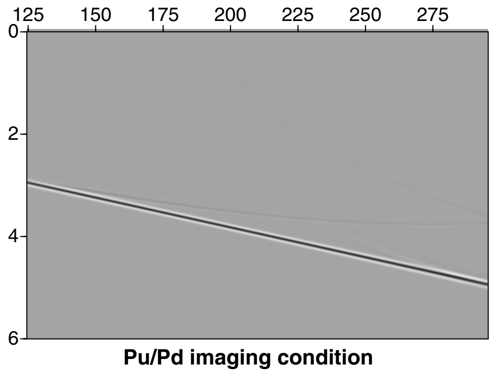

Figure 5. A true amplitude WEM image for the dipping planar reflector.

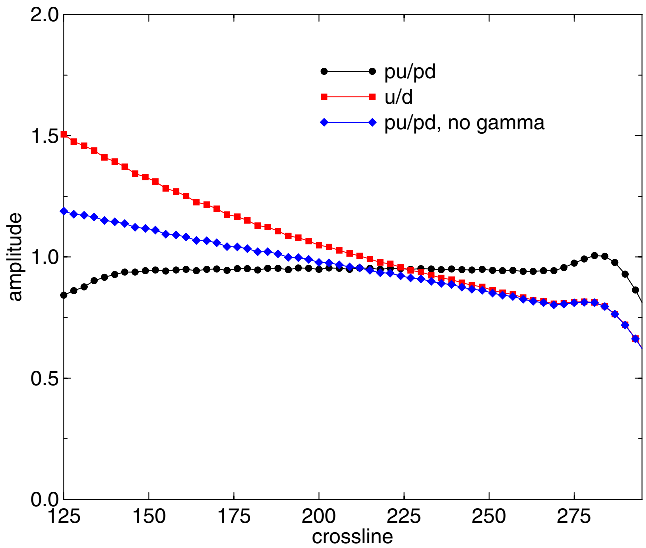

Figure 6. The peak amplitude along the migrated dipping reflector for three different WEMs. The curve highest on the left (red, in the colour rendition) is the output from Claerbout's WEM. The curve next higher on the left (blue, in the colour rendition) arises when we set $\Gamma = 0$ in (32) and (33). The curve lowest on the left (black, in the colour rendition) is the output of our true amplitude WEM.

To further confirm this, we provide a second example in which the plane is dipping to the right at 30°. The data for this example are shown in figure 4. Here, the ratio of amplitudes of the upgoing and downgoing Green functions is no longer equal to one at the reflector. Again, we produce the reflection via a density contrast with reflection coefficient equal to one. The output of our true amplitude WEM processing is shown in figure 5. The image for Claerbout WEM is the same.

In figure 6, we show peak amplitudes for three different WEMs. The curve highest on the left (red, in the colour rendition) is the output from Claerbout's WEM using Claerbout's boundary condition for the source wavelet, (29). The curve next higher on the left (blue, in the colour rendition) arises when we set $\Gamma = 0$ in (32) and (33). This is equivalent to using Claerbout's equations, but the source boundary condition of (32). The curve lowest on the left (black, in the colour rendition) is the output of our true amplitude WEM in (32) and (33).

It can be seen that, except for aperture affects, this curve is close to unity over the entire range of the reflector. Thus, this test compares the proposed method to Claerbout's method with and without the additional scaling by $(\mathrm{i}\Lambda)^{-1}$ in the boundary condition. It is clear that the true amplitude WEM produces the most accurate amplitude for this dipping plane example.

From these examples, we see that the true amplitude algorithm recovers the reflectivity accurately, aside from the edge effects and small jitters caused by interference with wraparound artifacts.

7. Conclusions

Common-shot migrations offer good potential for imaging complex structures, but the conventional formulations of such migrations produce incorrect migrated amplitudes. Here, we have described true amplitude one-way wave equations that allow us to extend the standard method both for forward modelling and for wave equation migration. These modified one-way wave operators are developed with the aid of pseudo-differential operator theory. We have provided proofs that these new one-way wave equations provide solutions that agree dynamically, as well as kinematically, with the solutions of the full wave equation. Further, we have proposed a new approach to WEM, transforming it into a true amplitude process, meaning that it produces an inversion output that agrees asymptotically with Kirchhoff inversion: it produces a reflector map with peak amplitudes on the reflector in known proportion to the geometrical optics reflection coefficient. We have provided a proof of this claim. With the aid of a simple numerical example, we demonstrated that the migration method that we proposed does calibrate common-shot migrations by correcting both their amplitude and phase behaviour for an example in which the wave speed is depth dependent—$v = v(z)$. The new method actually builds a bridge between true amplitude common-shot Kirchhoff migration and the migrations based on one-way wavefield extrapolation.

Acknowledgments

GQZ's research was partially supported by the Major State Basic Research Programme of the People's Republic of China (G1999032800). All three authors further acknowledge the support and approval for publication of Veritas DGC.

The authors also gratefully acknowledge the reviewers of the paper. By following their suggestions, we believe that we have significantly improved the exposition in this paper.

We also extend our gratitude to John Stockwell for help with the rendition of most of our figures.

Appendix A

In this appendix, we show that the one-way wave equations that we developed in this paper provided the same travel time and amplitudes as the full wave equation (1). More specifically, we start from the one-way wave equation for the downward propagating wave D defined by (37) in two spatial dimensions (to make some of the calculations simpler to follow). That equation is repeated here as

$$\left[\frac{\partial}{\partial z} + \Lambda\right] D(x, z; t) + \Gamma D = 0. \tag{A.1}$$

Motivated by the source adjustment introduced in (32), we introduce the downgoing wave of the inversion process through the scaling

$$p_D = \Lambda^{-1} D \quad \Leftrightarrow \quad \Lambda p_D = D. \tag{A.2}$$

Our objective is to prove that the eikonal equation of (A.1) is the downgoing branch of the eikonal equation for the full wave equation (1) in heterogeneous media. This equations will be specified below after we introduce our notation for this asymptotic analysis. Clearly, we could define a corresponding function p_U and carry out the same analysis for the upgoing wave. Below, we will see that the eikonal equation for the downgoing wave chooses the sign of the z-derivative of the travel time to agree with the sign of ω, guaranteeing downward propagation. For p_U, the sign of the z-derivative is opposite to the sign of ω and that is the only different in the analysis provided here.

We will then show that the leading order amplitude of p_D satisfies the same transport equation as the amplitude of the full wave equation. Further, the sign of the z-derivative will be a common multiplier of all terms of the derived transport equation, ensuring that the same will be true for the solution p_U.

We start from equation (27), specialized to downgoing waves in two spatial dimensions:

$$\frac{\partial D}{\partial z} + \frac{1}{v}\frac{\partial D}{\partial t} - \frac{1}{\pi v}\frac{\partial}{\partial t}\int_{-1}^{1}\sqrt{1-s^2}q_D(s;x,z,t)\,\mathrm{d}s + \frac{v_z}{2v}[D + q_D(1;x,z,t)] = 0. \tag{A.3}$$

Here, $D = D(x,z,t)$ and $q_D(s;\cdot)$ satisfies

$$\left\{\frac{\partial^2}{\partial t^2} - s^2\left(v\frac{\partial}{\partial x}\right)^2\right\}q_D(s;x,z;t) = \left(v\frac{\partial}{\partial x}\right)^2 D(x,z;t). \tag{A.4}$$

We use the definition of p_D in (A.2) and the definition (26) for Λ to write

$$\frac{1}{v}\frac{\partial}{\partial t}p_D - \frac{1}{v}\frac{\partial}{\partial t}\left(\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}q_p(s;x,z,t)\,\mathrm{d}s\right) = D(x,z;t), \tag{A.5}$$

where $q_p(s;\cdot)$ satisfies

$$\left\{\frac{\partial^2}{\partial t^2} - s^2\left(v\frac{\partial}{\partial x}\right)^2\right\}q_p(s;x,z;t) = \left(v\frac{\partial}{\partial x}\right)^2 p_D(x,z;t). \tag{A.6}$$

A.1. High frequency asymptotic expansion

We consider only the downgoing wave equation (A.1). For the upgoing wave equation, all results can be obtained by the same approach.

We seek the solution of (A.1) in the form of the following asymptotic expansion:

$$F(x,z;t) \sim \mathrm{e}^{\mathrm{i}\omega(t-\varphi(x,z))}\sum_{j=0} A^j(x,z)\omega^{-j}. \tag{A.7}$$

A^0 is simply written as $A(F)$ (F can be either D or q_D). We simply denote p_D by p, to be determined in the form

$$p(x,z;t) \sim \mathrm{e}^{\mathrm{i}\omega(t-\varphi(x,z))}\sum_{j=0} A^{j+1}\omega^{-(j+1)}. \tag{A.8}$$

By substituting these asymptotic expansions into (A.4), we find that

$$A(q(D))(s;\cdot) = \frac{v^2\varphi_x^2}{1-s^2v^2\varphi_x^2}A(D) = e(s)A(D), \tag{A.9}$$

with

$$e(s) = \frac{v^2\varphi_x^2}{1-s^2v^2\varphi_x^2}. \tag{A.10}$$

Now we substitute the asymptotic expansions into (A.5) and (A.6) to obtain

$$A(D) = E(D)A(p), \tag{A.11}$$

with

$$E(D) = \frac{1}{v}[1 - v^2\varphi_x^2]^{1/2}. \tag{A.12}$$

For $e(s)$ in (A.10), we will also have need of $\partial e(s)/\partial x$. That result is

$$\begin{aligned} e(s)_x &= \frac{\partial}{\partial x}\left(\frac{v^2\varphi_x^2}{1 - s^2v^2\varphi_x^2}\right) = -\frac{1}{s^2}\frac{\partial}{\partial x}\left(1 - \frac{1}{1 - s^2v^2\varphi_x^2}\right) \\ &= \frac{1}{s^2}\left[\frac{s^2(v^2\varphi_x^2)_x}{(1 - s^2v^2\varphi_x^2)^2}\right] = \frac{(v^2\varphi_x^2)_x}{(1 - s^2v^2\varphi_x^2)^2}. \end{aligned} \tag{A.13}$$

A.2. Asymptotic solution of the downgoing one-way wave equation

By substituting the equation (A.8) into equations (A.3) and (A.4), we have

$$\begin{aligned} &\mathrm{i}\omega\left[-\left(\varphi_z - \frac{1}{v}\right)A(D) - \frac{1}{v\pi}\int_{-1}^{1}\sqrt{1 - s^2}A(q_D(s;\cdot))\,\mathrm{d}s\right] \\ &\qquad + \left[A(D)_z + \frac{v_z}{2v}(A(D) + A(q(1;\cdot)))\right] + \frac{1}{\mathrm{i}\omega}[\cdots] + \cdots = 0, \end{aligned} \tag{A.14}$$

and

$$\begin{aligned} &-\omega^2[(1 - s^2v^2\varphi_x^2)A(q_D(s;\cdot)) - v^2\varphi_x^2A(D)] \\ &\qquad + \mathrm{i}\omega[s^2B(A(q_D(s;\cdot))) + B(A(D))] + [\cdots] + \cdots = 0, \end{aligned} \tag{A.15}$$

where

$$B(A) = 2v^2\varphi_xA_x + A\left(v\frac{\partial}{\partial x}\right)^2\varphi. \tag{A.16}$$

We will replace the integral operator on $A(q_D(s;\cdot))$ in (A.14) by the same type of operator on $A(D)$ itself, since the latter is independent of s. To do so, we integrate (A.15) on the interval $s \in (-1, 1)$ with the weight $\sqrt{1 - s^2}/(v\pi(1 - s^2v^2\varphi_x^2))$. Then, we add the result of that integration to (A.14) multiplied by $\mathrm{i}\omega$ to obtain the equation for the asymptotic solution of the equation for the downgoing one-way wave:

$$\begin{aligned} &-\omega^2\left[-\left(\varphi_z - \frac{1}{v}\right)A(D) - \frac{1}{v\pi}\int_{-1}^{1}\sqrt{1 - s^2}\frac{v^2\varphi_x^2\,\mathrm{d}s}{1 - s^2v^2\varphi_x^2}A(D)\right] \\ &\qquad + \mathrm{i}\omega\left[A(D)_z + \frac{v_z}{2v}(A(D) + A(q(1;\cdot))) \right.\\ &\qquad \left.+ \frac{1}{v\pi}\int_{-1}^{1}\sqrt{1 - s^2}\frac{B(A(D)) + s^2B(A(q_D(s;\cdot)))}{1 - s^2v^2\varphi_x^2}\,\mathrm{d}s\right] + [\cdots] + \cdots = 0. \end{aligned} \tag{A.17}$$

The following integrals are needed:

$$J_n(b^2) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1 - s^2}\,\frac{\mathrm{d}s}{(1 - b^2s^2)^n}. \tag{A.18}$$

We have

$$J_0 = \tfrac{1}{2}, \qquad J_1(b^2) = \frac{1 - (1 - b^2)^{1/2}}{b^2},$$

$$J_2(b^2) = \frac{1}{2(1 - b^2)^{1/2}} \qquad \text{and} \qquad J_3(b^2) = \frac{3(1 - b^2) + 1}{8(1 - b^2)^{3/2}}.$$

Then we can obtain the following integrals:

$$I_0 = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}e(s)\,\mathrm{d}s = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{v^2\varphi_x^2\,\mathrm{d}s}{1-s^2v^2\varphi_x^2} = v^2\varphi_x^2 J_1(v^2\varphi_x^2), \tag{A.19}$$

$$I_1 = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{s^2e(s)\,\mathrm{d}s}{1-s^2v^2\varphi_x^2} = J_2(v^2\varphi_x^2) - J_1(v^2\varphi_x^2), \tag{A.20}$$

and

$$\begin{aligned} I_2 &= \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{s^2e(s)_x\,\mathrm{d}s}{1-s^2v^2\varphi_x^2} = \frac{(v^2\varphi_x^2)_x}{v^2\varphi_x^2}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{v^2\varphi_x^2\,\mathrm{d}s}{(1-s^2v^2\varphi_x^2)^3}\right] \\ &= \frac{(v^2\varphi_x^2)_x}{v^2\varphi_x^2}[J_3(v^2\varphi_x^2) - J_2(v^2\varphi_x^2)] = \frac{(v^2\varphi_x^2)_x}{8(1-v^2\varphi_x^2)^{3/2}}, \end{aligned} \tag{A.21}$$

with $e(s)$ defined by (A.10).

Now we can proceed to simplify the various terms in (A.15). According to the definition of $B(A)$ in (A.16) and using (A.11), we have

$$\begin{aligned} B(A(D)) &= 2v^2\varphi_x A(D)_x + A(D)\left(v\frac{\partial}{\partial x}\right)^2\varphi \\ &= 2v^2\varphi_x(E(D)A(p))_x + E(D)A(p)\left(v\frac{\partial}{\partial x}\right)^2\varphi \\ &= E(D)B(A(p)) + 2v^2\varphi_x E(D)_x A(p). \end{aligned} \tag{A.22}$$

Similarly, using (A.2), we find that

$$B(A(q_D(s;\cdot))) = e(s)E(D)B(A(p)) + 2v^2\varphi_x(e(s)E(D))_x A(p). \tag{A.23}$$

So, for the order-iω term in (A.15) we conclude that

$$B(A(D)) + s^2B(A(q_D)) = E(D)B(A(p))(1+s^2e(s)) + 2v^2\varphi_x A(p)[(1+s^2e(s))E(D)]_x. \tag{A.24}$$

Now we can carry out the integration in the order-iω term in (A.17), using all of the results (A.19)–(A.22). The result is

$$\begin{aligned} &\frac{1}{v\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{B(A(D)) + s^2B(A(q_D(s;\cdot)))}{1-s^2v^2\varphi_x^2}\,\mathrm{d}s \\ &\quad = \frac{1}{v}\left\{E(D)\left[B(A(p))\left(\frac{I_0}{v^2\varphi_x^2}+I_1\right)\right.\right. \\ &\qquad \left.\left. + 2v^2\varphi_x A(p)I_2\right] + 2v^2\varphi_x E(D)_x A(p)\left[\frac{I_0}{v^2\varphi_x^2}+I_1\right]\right\} \\ &\quad = \frac{1}{v}\{E(D)[B(A(p))J_2(v^2\varphi_x^2) + 2v^2\varphi_x A(p)I_2] + 2v^2\varphi_x E(D)_x A(p)J_2(v^2\varphi_x^2)\}. \end{aligned} \tag{A.25}$$

A.3. Eikonal equation and transport equation for the restored downgoing wave

From the coefficient of ω^2 in (A.17) we have

$$\left[-\varphi_z + \frac{1}{v}\left(1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{v^2\varphi_x^2}{1-s^2v^2\varphi_x^2}\,\mathrm{d}s\right)\right]A(D) = 0. \tag{A.26}$$

We seek a nontrivial asymptotic solution, so $A(D) \neq 0$. Therefore we conclude that

$$\left[-\varphi_z + \frac{1}{v}\left(1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{v^2\varphi_x^2}{1-s^2v^2\varphi_x^2}\,\mathrm{d}s\right)\right] = \left[-\varphi_z + \frac{1}{v}[1 - v^2\varphi_x^2 J_1(v^2\varphi_x^2)]\right]$$
$$= -\varphi_z + \frac{1}{v}(1 - v^2\varphi_x^2)^{1/2} = 0. \qquad \text{(A.27)}$$

This is the eikonal equation of the downgoing one-way wave equation (A.1). We can see that this eikonal equation (A.27) is just the one of two branches of the eikonal equation for the full wave equation (1) in 2D, namely,

$$\varphi_x^2 + \varphi_z^2 = \frac{1}{v^2}. \qquad \text{(A.28)}$$

Clearly, except for details of computation, the derivation in 3D would follow along the same lines.

By setting the coefficient of the order-iω term in (A.17) we obtain the transport equation of the restored downgoing wave $A(p)$. We consider two parts in this coefficient. One part is an integral (A.25), restated and expanded upon here:

$$\begin{aligned}
I &= \frac{1}{v\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{B(A(D)) + s^2B(A(q_D(s;\cdot)))}{1-s^2v^2\varphi_x^2}\,\mathrm{d}s \\
&= \frac{E(D)}{v}\left\{\left[2v^2\varphi_x A(p)_x + A(p)\left(v\frac{\partial}{\partial x}\right)^2\varphi\right]J_2(v^2\varphi_x^2) + 2v^2\varphi_x A(p)I_2\right\} \\
&\quad + 2v\varphi_x E(D)_x A(p)J_2(v^2\varphi_x^2) \\
&= 2v\varphi_x E(D)A(p)_x J_2(v^2\varphi_x^2) \\
&\quad + \left\{\frac{E(D)}{v}\left[\left(v\frac{\partial}{\partial x}\right)^2\varphi J_2(v^2\varphi_x^2) + 2v^2\varphi_x I_2\right]\right. \\
&\quad \left. + 2v\varphi_x E(D)_x J_2(v^2\varphi_x^2)\right\}A(p). \qquad \text{(A.29)}
\end{aligned}$$

From (A.27), (A.12) and (A.21),

$$2vE(D)J_2(v^2\varphi_x^2) = 2\sqrt{1-v^2\varphi_x^2}J_2(v^2\varphi_x^2) = 1, \qquad \text{(A.30)}$$

$$\frac{E(D)}{v}2v^2\varphi_x^2 I_2 = 2vE(D)\varphi_x\frac{(v^2\varphi_x^2)_x}{8(1-v^2\varphi_x^2)^{3/2}} = \frac{\varphi_x(v^2\varphi_x^2)_x}{4(1-v^2\varphi_x^2)}, \qquad \text{(A.31)}$$

$$2v\varphi_x E(D)_x J_2(v^2\varphi_x^2) = \frac{v\varphi_x E(D)_x}{\sqrt{1-v^2\varphi_x^2}} = \frac{v\varphi_x}{2\sqrt{1-v^2\varphi_x^2}}\left[\varphi_{zx} - \frac{v\varphi_x\varphi_{xx} + v_x/v^2}{\sqrt{1-v^2\varphi_x^2}}\right]. \qquad \text{(A.32)}$$

Here, we derive $E(D)_x$ from (A.6):

$$E(D)_x = \frac{1}{2}\left[\varphi_z + \frac{1}{v}\sqrt{1-v^2\varphi_x^2}\right]_x = \frac{1}{2}\left[\varphi_{zx} - \frac{v\varphi_x\varphi_{xx} + v_x/v^2}{\sqrt{1-v^2\varphi_x^2}}\right].$$

By using these results, we can now simplify I in (A.29) as follows:

$$\begin{aligned}
\mathrm{I} &= \varphi_x A(p)_x + A(p)\left\{\frac{\left(v\frac{\partial}{\partial x}\right)^2\varphi}{2v^2} + \frac{\varphi_x(v^2\varphi_x^2)_x}{4(1-v^2\varphi_x^2)}\right. \\
&\quad \left. + \frac{v\varphi_x}{2\sqrt{1-v^2\varphi_x^2}}\left[\varphi_{zx} - \frac{v\varphi_x\varphi_{xx} + v_x/v^2}{\sqrt{1-v^2\varphi_x^2}}\right]\right\} \\
&= \varphi_x A(p)_x + A(p)\left\{\frac{1}{2v^2}(vv_x\varphi_x + v^2\varphi_{xx}) + \frac{v\varphi_x^2(v_x\varphi_x + v\varphi_{xx})}{2(1-v^2\varphi_x^2)}\right.
\end{aligned}$$

$$+ \frac{v\varphi_x}{2\sqrt{1-v^2\varphi_x^2}}\left[\varphi_{zx} - \frac{v\varphi_x\varphi_{xx} + v_x/v^2}{\sqrt{1-v^2\varphi_x^2}}\right]\Bigg\}$$

$$= \varphi_x A(p)_x + \frac{A(p)}{2}\Bigg\{\varphi_{xx} + \frac{v_x\varphi_x}{v} + \frac{v_x\varphi_x v^2\varphi_x^2}{v(1-v^2\varphi_x^2)}$$

$$+ \frac{v\varphi_x\varphi_{zx}}{\sqrt{1-v^2\varphi_x^2}} - \frac{v_x\varphi_x}{v(1-v^2\varphi_x^2)}\Bigg\}$$

$$= \varphi_x A(p)_x + \frac{A(p)}{2}\left[\varphi_{xx} + \frac{v\varphi_x\varphi_{zx}}{\sqrt{1-v^2\varphi_x^2}}\right]. \tag{A.33}$$

Another part in the coefficient of term iω in (A.17) is

$$\mathrm{II} = E(D)_z + \frac{v_z}{2v}(A(D) + A(q(1;\cdot))). \tag{A.34}$$

From (A.5), (A.6), (A.27) and (A.2) we have

$$\mathrm{II} = E(D)A(p)_z + E(D)_z A(p) + \frac{v_z}{2v}\frac{E(D)A(p)}{1-v^2\varphi_x^2}$$

$$= \varphi_z A(p)_z + A(p)\left[E(D)_z + \frac{v_z E(D)}{2v(1-v^2\varphi_x^2)}\right]$$

$$= \varphi_z A(p)_z + \frac{A(p)}{2}\left[\left(\varphi_z + \frac{1}{v}\sqrt{1-v^2\varphi_x^2}\right)_z + \frac{v_z}{v}\frac{\sqrt{1-v^2\varphi_x^2}}{v(1-v^2\varphi_x^2)}\right]$$

$$= \varphi_z A(p)_z + \frac{A(p)}{2}\left[\varphi_{zz} - \frac{v_z/v^2 + v\varphi_x\varphi_{xz}}{\sqrt{1-v^2\varphi_x^2}} + \frac{v_z/v^2}{\sqrt{1-v^2\varphi_x^2}}\right]$$

$$= \varphi_z A(p)_z + \frac{A(p)}{2}\left[\varphi_{zz} - \frac{v\varphi_x\varphi_{xz}}{\sqrt{1-v^2\varphi_x^2}}\right]. \tag{A.35}$$

Consequently the coefficient of the term iω in (A.17) is

$$\mathrm{I} + \mathrm{II} = \varphi_x A(p)_x + \frac{A(p)}{2}\left[\varphi_{xx} + \frac{v\varphi_x\varphi_{xz}}{\sqrt{1-v^2\varphi_x^2}}\right]$$

$$+ \varphi_z A(p)_z + \frac{A(p)}{2}\left[\varphi_{zz} - \frac{v\varphi_x\varphi_{xz}}{\sqrt{1-v^2\varphi_x^2}}\right]$$

$$= \nabla\varphi\cdot\nabla A(p) + \frac{A(p)}{2}\Delta\varphi. \tag{A.36}$$

This last expression, then, is the coefficient of the order-iω term in (A.17) and, therefore, must be equal to zero. This leads to the first transport equation for the amplitude of the downgoing wave; that is,

$$2\nabla\varphi\cdot\nabla A(p_D) + A(p_D)\Delta\varphi = 0. \tag{A.37}$$

This is just the transport equation for the full wave equation (1). Because φ is the downgoing travel time, the solution will be the amplitude for the downgoing wave.

A similar result can be derived for the upgoing one-way wave equation, starting again from (27), but using the lower signs. In that case, we would find that

$$\varphi_z + \frac{1}{v}\sqrt{1-v^2\varphi_x^2} = 0. \tag{A.38}$$

It is another branch of the eikonal equation (A.28) of the full wave equation (1). In the same manner, we can obtain the same transport equation for the first amplitude of the restored upcoming wave $A(p_U)$:

$$2\nabla\varphi\cdot\nabla A(p_U) + A(p_U)\Delta\varphi = 0. \tag{A.39}$$

This completes the proof.

Appendix B

In this appendix, we verify the integral identity (22) for $\lambda = \mathrm{i}k_z$. More to the point, let us consider the integral

$$I = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,\mathrm{d}s. \tag{B.1}$$

We view s as a complex variable and replace the 'contour' from -1 to 1 by the 'barbell' contour of figure B.1 extending from $-1+\epsilon$ to $1-\epsilon$, then passing around the branch point at $s = 1$ in a clockwise direction, returning to $-1+\epsilon$ encircling the branch point at $s = -1$ in a clockwise direction to complete a closed path of integration. The square root changes sign when the path passes around the branch point at either end. Thus, when passing around both branch points, the integrand returns to its original value; this justifies the claim that the contour of integration is closed. On the other hand, after one change of sign passing around the branch point at $1-\epsilon$, the integrand has changed sign compared to its previous value at each s. However, the direction of the path of integration has reversed as well. Thus, the integral on the path approximately between $1-\epsilon$ and $-1+\epsilon$ after that first circumnavigation is the same as the integral before. Further, it is standard in complex integration methodology to confirm that the integrals on the circles of radius ϵ shrink to zero as $\epsilon \to 0$. Thus, calling this new contour of integration C_1, we need only introduce a factor of $1/2$ to equate the integral on C_1 to the original real integral on the interval $(-1, 1)$:

$$I = \frac{1}{2\pi}\int_{C_1}\sqrt{1-s^2}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,\mathrm{d}s. \tag{B.2}$$

We note that the integrand has two poles at $s_\pm = \pm|\omega|/(v|\bar{k}|) > 1$. We propose to recast the integral as a sum of residues at these poles plus an integral on a circle of radius $r > |\omega|/(v|\bar{k}|)$. Since we will now be concerned with the region where $|s| > 1$, we prefer to rewrite the integrand as

$$\sqrt{1-s^2}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2} = -\mathrm{i}\sqrt{s^2-1}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2} \tag{B.3}$$

with

$$\sqrt{s^2-1} \approx s$$

for $|s|$ 'large'. To confirm this, note that we passed over the branch point at $s = 1$ to pass to the region of $|s| > 1$. In doing so, $\arg(1-s)$ passed from zero to $-\pi$, in which case the argument of this square root passed from zero to $-\pi/2$, which is the argument of $-\mathrm{i}$; hence the choice of sign in the redefined square root.

We are now prepared to recast I as an integral on the contour of radius r plus residues, namely, the integral on the path C_2 of figure B.1:

$$\begin{aligned} I &= -\frac{\mathrm{i}}{2\pi}\int_{C_2}\sqrt{s^2-1}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,\mathrm{d}s \\ &= \frac{1}{2\pi\mathrm{i}}\int_{C_2}\sqrt{s^2-1}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,\mathrm{d}s \\ &= \sum_{\pm}(\text{Residues}, s = s_\pm) - \frac{1}{2\pi\mathrm{i}}\int_{|s|=r}\sqrt{s^2-1}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,\mathrm{d}s. \end{aligned} \tag{B.4}$$

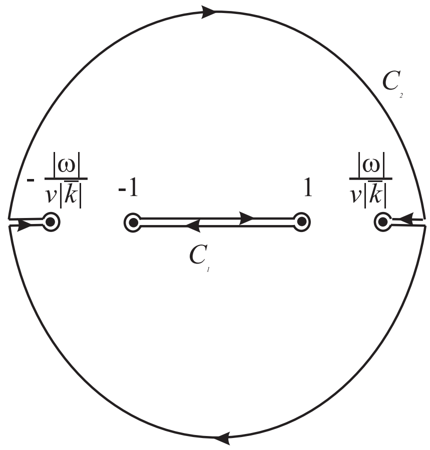

Figure B.1. Contours of integration. C_1 is the barbell contour. Deformation outward leads to the contour C_2.

In the last line here, we have observed that the deformation of the contour onto the circle leads to two anticlockwise contours around the poles. Further, in the last integral, we have reverted to the default that an integral on a contour of prescribed radius is understood to be an anticlockwise contour; hence, the change in sign on the integral from the sign in the previous line, where the direction of C_1 was clockwise and the direction of C_2 is clockwise.

Now,

$$\sum_{\pm}(\text{Residues}, s = s_\pm) = \sum_{\pm} \sqrt{s^2-1}\frac{(v\bar{k})^2}{-2s(v\bar{k})^2}\bigg|_{s=s_\pm}. \tag{B.5}$$

Both the square root here and s itself change sign in the evaluation at $s_\pm$. Consequently, these two terms add to yield

$$\sum_{\pm}(\text{Residues}, s = s_\pm) = -\sqrt{1-\frac{(v\bar{k})^2}{\omega^2}}. \tag{B.6}$$

Next, we must evaluate the integral over $|s| = r$ in the limit as $r \to \infty$. For large r, use

$$\sqrt{s^2-1} \approx s \qquad \text{and} \qquad \frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2} \approx -\frac{1}{s^2}$$

to obtain

$$\begin{aligned} -\frac{1}{2\pi \mathrm{i}}\int_{|s|=r}\sqrt{s^2-1}\frac{(v\bar{k})^2}{\omega^2-s^2(v\bar{k})^2}\,\mathrm{d}s &\approx \frac{1}{2\pi \mathrm{i}}\int_{|s|=r}\frac{\mathrm{d}s}{s} \\ &= \frac{1}{2\pi \mathrm{i}}\int_0^{2\pi} \mathrm{i}\,\mathrm{d}\theta = 1, \qquad s = r\mathrm{e}^{\mathrm{i}\theta}, \quad \mathrm{d}s = \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta. \end{aligned} \tag{B.7}$$

Combining this result with (B.6) in (B.4) we conclude that

$$I = 1 - \sqrt{1-\frac{(v\bar{k})^2}{\omega^2}}. \tag{B.8}$$

We only need to substitute this result into (22) to confirm that identity, which is the first result that we set out to verify in this appendix.

Next, we turn to the issue of a corresponding representation for λ^{-1} in order to provide an interpretation for the pseudo-differential operator, Λ^{-1}.

Let

$$\xi = \frac{v\bar{k}}{\omega}, \tag{B.9}$$

and observe that

$$\begin{aligned}\frac{\mathrm{i}\omega}{v}\lambda^{-1} &= \frac{1}{\sqrt{1-\xi^2}} = \frac{\sqrt{1-\xi^2}}{1-\xi^2} = \frac{1}{1-\xi^2}\left(1 - \frac{1}{\pi}\int_{-1}^{1}\frac{\xi^2\sqrt{1-s^2}}{1-\xi^2 s^2}\,\mathrm{d}s\right)\\ &= 1 + \frac{\xi^2}{1-\xi^2}\left(1 - \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}}{1-\xi^2 s^2}\,\mathrm{d}s\right).\end{aligned} \tag{B.10}$$

The last result on the first line here follows from (22), confirmed above. The last line arises from writing

$$\frac{1}{1-\xi^2} = \frac{1-\xi^2+\xi^2}{1-\xi^2} = 1 + \frac{\xi^2}{1-\xi^2}$$

and re-collecting terms in the previous result.

Since

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\mathrm{d}s}{\sqrt{1-s^2}} = 1,$$

we have

$$\begin{aligned}\frac{\mathrm{i}\omega}{v}\lambda^{-1} &= 1 + \frac{\xi^2}{1-\xi^2}\frac{1}{\pi}\int_{-1}^{1}\left(\frac{1}{\sqrt{1-s^2}} - \frac{\sqrt{1-s^2}}{1-\xi^2 s^2}\right)\mathrm{d}s\\ &= 1 + \frac{1}{\pi}\int_{-1}^{1}\frac{\xi^2}{1-\xi^2}\frac{1}{\sqrt{1-s^2}}\left(1 - \frac{1-s^2}{1-\xi^2 s^2}\right)\mathrm{d}s\\ &= 1 + \frac{1}{\pi}\int_{-1}^{1}\frac{\xi^2}{\sqrt{1-s^2}}\frac{s^2}{1-\xi^2 s^2}\,\mathrm{d}s\\ &= 1 + \frac{1}{\pi}\int_{-1}^{1}\frac{s^2}{\sqrt{1-s^2}}\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,\mathrm{d}s.\end{aligned} \tag{B.11}$$

Now, in analogy to (24), we find that

$$\frac{1}{v}\frac{\partial}{\partial t}\Lambda^{-1} = \left\{I + \frac{1}{\pi}\int_{-1}^{1}\frac{s}{\sqrt{1-s^2}}\mathrm{L}_T^{-1}(s;\vec{\rho},z,t)(v\nabla_{Tx})^2\,\mathrm{d}s\right\}. \tag{B.12}$$

Appendix C

In this appendix, we derive the representation (39) for p_U. Because of the proof of appendix A, we can use the full wave equation to find p_U. That is, p_U must satisfy the frequency domain equation

$$\mathcal{L}_\omega p_U(\vec{x};\omega) = \nabla^2 p_U + \frac{\omega^2}{v^2}p_U = 0, \qquad p_U(x,y,0;\omega) = Q(x,y;\omega). \tag{C.1}$$

Furthermore, p_U is an upward propagating wave and therefore it satisfies the *inward propagating Sommerfeld radiation conditions* (Bleistein 1984),

$$rp_U(\vec{x};\omega)\ \text{bounded}, \qquad r\left[\frac{\partial p_U}{\partial r} - \frac{\mathrm{i}\omega}{v}p_U\right] \to 0, \quad r\to\infty, \quad r = |\vec{x}|. \tag{C.2}$$

We introduce a Green function, $G(\vec{x}, \vec{x}'; \omega)$ satisfying the following problem:

$$\mathcal{L}_\omega G = -\delta(\vec{x} - \vec{x}'), \qquad G(x, y, 0, \vec{x}') = 0. \tag{C.3}$$

We also require that G satisfy the same radiation condition as p_U, namely

$$rG(\vec{x}, \vec{x}'; \omega) \text{ bounded}, \qquad r\left[\frac{\partial G}{\partial r} - \frac{\mathrm{i}\omega}{v} G\right] \to 0. \tag{C.4}$$

Now, we consider the integral

$$I = \int_D \{p_U(\vec{x}; \omega)\mathcal{L}_\omega G(\vec{x}, \vec{x}'; \omega) - G(\vec{x}, \vec{x}'; \omega)\mathcal{L}_\omega p_U(\vec{x}; \omega)\}\, \mathrm{d}^3x. \tag{C.5}$$

Here, D is a hemisphere of radius R centred at the origin, planar boundary at $z = 0$ and extending down into the domain $z > 0$.

We use the wave equations in (C.1) and (C.3) to replace the differential operators in the integral in (C.5) by the source terms and then conclude that

$$I = -p_U(\vec{x}'; \omega). \tag{C.6}$$

Now, we note that

$$\begin{aligned} &p_U(\vec{x}; \omega)\mathcal{L}_\omega G(\vec{x}, \vec{x}'; \omega) - G(\vec{x}, \vec{x}'; \omega)\mathcal{L}_\omega p_U(\vec{x}; \omega) \\ &\qquad = p_U(\vec{x}; \omega)\nabla^2 G(\vec{x}, \vec{x}'; \omega) - G(\vec{x}, \vec{x}'; \omega)\nabla^2 p_U(\vec{x}; \omega). \end{aligned}$$

Next, we use Green theorem (Bleistein 1984) to recast the integral in (C.5) as one over the boundary ∂D of the domain D:

$$I = \int_{\partial D} \left\{p_U(\vec{x}; \omega)\frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial n} - G(\vec{x}, \vec{x}'; \omega)\frac{\partial p_U(\vec{x}; \omega)}{\partial n}\right\} \mathrm{d}S. \tag{C.7}$$

Here, $\partial/\partial n$ is the normal derivative in the outward direction from ∂D. This surface consists of two pieces.

First, there is the hemisphere of radius R on which $\partial/\partial n = -\partial/\partial r$. The second part of ∂D is the disc on the plane at $z = 0$ and of radius R, centred at the origin. On this disc, $\partial/\partial n = -\partial/\partial z$. On the hemisphere, we write

$$\begin{aligned} \Delta &= p_U(\vec{x}; \omega)\frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial n} - G(\vec{x}, \vec{x}'; \omega)\frac{\partial p_U(\vec{x}; \omega)}{\partial n} \\ &= p_U(\vec{x}; \omega)\frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial r} - G(\vec{x}, \vec{x}'; \omega)\frac{\partial p_U(\vec{x}; \omega)}{\partial r} \\ &= p_U(\vec{x}; \omega)\left\{\frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial r} - \frac{\mathrm{i}\omega}{v}G(\vec{x}, \vec{x}'; \omega)\right\} \\ &\quad - G(\vec{x}, \vec{x}'; \omega)\left\{\frac{\partial p_U(\vec{x}; \omega)}{\partial r} - \frac{\mathrm{i}\omega}{v}p_U\right\}. \end{aligned}$$

Note that we can write $\mathrm{d}S = R^2\, \mathrm{d}\Omega$ in (C.7) for this hemisphere. Here, $\mathrm{d}\Omega$ is the differential solid angle on the unit sphere. We need to evaluate the last expression at $r = R$. We can pair up a multiplier of R from the $\mathrm{d}S$ multiplication with each of the factors in this last expression for Δ. Using the Sommerfeld conditions in (C.2) we conclude that

$$\begin{aligned} &Rp_U(\vec{x}; \omega)R\left\{\frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial r} - \frac{\mathrm{i}\omega}{v}G(\vec{x}, \vec{x}'; \omega)\right\}\Bigg|_{r=R} \to 0, \qquad R \to \infty \\ &RG(\vec{x}, \vec{x}'; \omega)R\left\{\frac{\partial p_U(\vec{x}; \omega)}{\partial r} - \frac{\mathrm{i}\omega}{v}p_U\right\}\Bigg|_{r=R} \to 0, \qquad R \to \infty. \end{aligned}$$

Since the domain of integration is bounded (by 2π), we conclude that the surface integral in (C.7) over the hemisphere must approach zero as the radius of the hemisphere approaches infinity. Consequently, we are left with the surface integral over the plane $z = 0$ in that integral.

Here, we use the boundary data in (C.1) and (C.3) plus the fact that $\partial/\partial n = -\partial/\partial z$ to conclude that

$$p_U(\vec{x}'; \omega) = \int_{z=0} Q(x, y; \omega) \frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial z} \, \mathrm{d}x \, \mathrm{d}y. \tag{C.8}$$

Now, we have to examine the Green function G more closely. Let us introduce the free space Green function, $G_s(\vec{x}, \vec{x}'; \omega)$, satisfying the same radiation conditions as G. Further, let us introduce the *image point* $\vec{x}^* = (x, y, z)$. We claim that the solution G can then be constructed by the 'method of images' as

$$G(\vec{x}, \vec{x}'; \omega) = G_s(\vec{x}, \vec{x}'; \omega) - G_s(\vec{x}^*, \vec{x}'; \omega). \tag{C.9}$$

At $z = 0$, this solution is zero as required. Now, using ray theory, we can write

$$G(\vec{x}, \vec{x}'; \omega) = A(\vec{x}, \vec{x}')\mathrm{e}^{\mathrm{i}\omega\varphi(\vec{x}, \vec{x}')} - A(\vec{x}^*, \vec{x}')\mathrm{e}^{\mathrm{i}\omega\varphi(\vec{x}^*, \vec{x}')}.$$

To leading order the z-derivative requires differentiation of the phases, only. In those phases, the z-derivatives at $z = 0$ have opposite sign but are otherwise equal. Therefore, we conclude that

$$\left. \frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial z} \right|_{z=0} \sim 2\mathrm{i}\omega \frac{\partial \varphi}{\partial z} A(\vec{x}, \vec{x}')\mathrm{e}^{\mathrm{i}\omega\varphi(\vec{x}, \vec{x}')} \bigg|_{z=0}. \tag{C.10}$$

Now we use this result and (C.9) in (C.8) to conclude that

$$p_U(\vec{x}'; \omega) = 2\mathrm{i}\omega \int_{z=0} Q(x, y; \omega) \frac{\partial \varphi}{\partial z} A(\vec{x}, \vec{x}')\mathrm{e}^{\mathrm{i}\omega\varphi(\vec{x}, \vec{x}')} \bigg|_{z=0} \mathrm{d}x \, \mathrm{d}y. \tag{C.11}$$

To complete the story, we need to replace $\vec{x}'$ by $\vec{x}$ and replace $\vec{x}$ at $z = 0$ by $\vec{x}_r$. Furthermore, we use the fact that $\partial\varphi/\partial z$ is the z-component of the travel time φ, which is a vector making the angle α_r with the vertical and having magnitude $1/v(\vec{x}_r)$. The result of these substitutions is equation (39). This completes the verification of that equation.

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