Galerkin-wavelet modeling of wave propagation: Optimal finite-difference stencil design

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Abstract

In this paper we describe a method for design of optimal finite-difference stencils for wave propagation problems using an intrinsically explicit Galerkin-wavelet formulation. The method enables an efficient choice of stencils optimal for a certain problem. We compare group velocity curves corresponding to stencils obtained by our choice of wavelet basis and traditional finite-difference schemes. Generally there exist choices of stencils with superior characteristics compared to conventional finite-difference stencils of the same size. Beside gain in accuracy, this leads to large computational savings.

1 Introduction

In this paper we propose an effective method for designing finite-difference stencils optimal for specific wave propagation problems. Compactly supported orthonormal wavelets [5, 6] have been proposed as a tool for numerical solution of PDE's [7]. This paper examines a family of Galerkin approximations to the wave equation, using compactly supported wavelets as basis functions. We observe immediately that the orthonormality of the wavelet basis makes a Galerkin approximation into an explicit finite-difference scheme. In contrast, standard finite-element methods for hyperbolic equations are naturally implicit and require mass matrix lumping to produce explicit schemes [20, 4]. Hence the simplicity and computational inexpensive nature of explicit finite-differences are maintained while several new possibilities open up.

Holberg [10] optimized the coefficients of finite-difference stencils to yield optimal numerical dispersion over a prescribed band. The general optimization problem is ill conditioned. The space of stencils corresponding to wavelets is a subset of all possible stencils, in which the optimization problem is much more well conditioned.

2 Wavelet basis functions

Wavelets are sets of functions that are concentrated or localized in time or space and are generated from a basic or mother function by translation and scaling [2, 3, 5, 6]. The wavelets generated from a basic mother or *scaling function*, $\varphi(x)$, which in order to have so-called multi-resolution properties [13, 14], must satisfy

$$\varphi(x) = \sum_{k=0}^{N-1} h_k \varphi(2x-k). \tag{2.1}$$

The basic or mother wavelet is defined in terms of this scaling functions by

$$\psi(x) = \sum_{k=0}^{N-1} g_k \varphi(2x-k), \tag{2.2}$$

where

$$g_k = (-1)^k h_{N-1-k}. \tag{2.3}$$

From these basic functions, a family of scaling functions is generated by translation from

$$\varphi_k(x) = \varphi(x-k), \tag{2.4}$$

and a family of wavelets by translation and scaling from,

$$\psi_{jk}(x) = \psi(2^j x - k). \tag{2.5}$$

In addition to the multi-resolution properties assumed by Daubechies [5], if we require that integer translates of the scaling function and integer translates and scaling of the wavelets be orthonormal, and further, that the support of the basic functions be finite, one can show [2, 5] that the coefficients in equation (2.1) must satisfy the equation

$$\sum_k h_k = 2, \tag{2.6}$$

as well as the N/2 equations

$$\sum_k h_k h_{k-2n} = \begin{cases} 2 & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

If the N coefficients satisfy equations (2.6) and (2.7), that uses $N/2+1$ degrees of freedom leaving $N/2-1$ for optimization [9, 17]. Indeed one of the features that distinguishes wavelet systems from most other orthonormal basis sets is having parameters to tailor the system to a particular problem. A general parameterization has been given in [2, 18, 21]. Daubechies [5, 6] used these to maximize the number of vanishing moments. For a length-6 coefficient sequence, $h(n)$, the two parameters are defined as α and β and the resulting coefficients are,

$$\begin{aligned}
h(0) &= [(1+cos(\alpha)+sin(\alpha))(1-cos(\beta)-sin(\beta))+2sin(\beta)cos(\alpha)]/4\\
h(1) &= [(1-cos(\alpha)+sin(\alpha))(1+cos(\beta)-sin(\beta))-2sin(\beta)cos(\alpha)]/4\\
h(2) &= [1+cos(\alpha-\beta)+sin(\alpha-\beta)]/2\\
h(3) &= [1+cos(\alpha-\beta)-sin(\alpha-\beta)]/2\\
h(4) &= 1-h(0)-h(2)\\
h(5) &= 1-h(1)-h(3).
\end{aligned} \tag{2.8}$$

This parameterization suits our purpose of finding wavelets with optimal dispersion characteristics well. 4.2 and Figures 2.1a-d show examples of wavelets with the corresponding α-β parameters. These figures were generated using the successive approximation technique of wavelet scaling functions described in Burrus and Gopinath [2].

3 Galerkin-wavelet formulation of a system of first order PDE's

On a general form a semi-discrete Galerkin approximation to a first order PDE may be written as,

$$M\dot{u} = Kv + f, \tag{3.1}$$

where M is the mass matrix, K the stiffness matrix, and f a source vector [19]. The orthonormality of the wavelet basis reduces M to the identity matrix. The compact support makes K asymmetric and banded with integrals of the scaling functions times discrete spatial translations of the first derivative of it as elements. These are called connection coefficients [12] and will be denoted by the vector $\Gamma = \{\Gamma^j\}$. Here j denotes the discrete distance between the two functions. Fortunately there exists a very efficient and accurate algorithm to compute the connection coefficients [1].

The Mallat algorithm [13] allows us to derive explicit approximations to derivatives of functions expressed in a basis of orthonormal wavelets with compact support. McCormick and Wells [15] define the wavelet discrete differentiation operator as,

$$D_J v(x_l) = \sum_m \frac{1}{h_J}\Gamma^{l-m} v(x_m), \tag{3.2}$$

where $h_J = 2^{-J}$ for wavelet bases with multiplicity 2. The wavelet approximation thus reduces to something which in a computational sense is very similar to a finite-difference stencil. As a matter of fact it is easy to show that for $N \leq 4$, the Daubechies wavelet connection coefficients (D2 and D4 wavelets) are identical to centered finite-difference stencil coefficients. To integrate the time derivative, in equation (3.1), in time we used the Leap-frog scheme.

4 Finite-difference stencil design

The α-β-parameterization allows us to find wavelets, which are optimal, in the sense that they yield a numerical group velocity close to the analytical group velocity for a given wave propagation

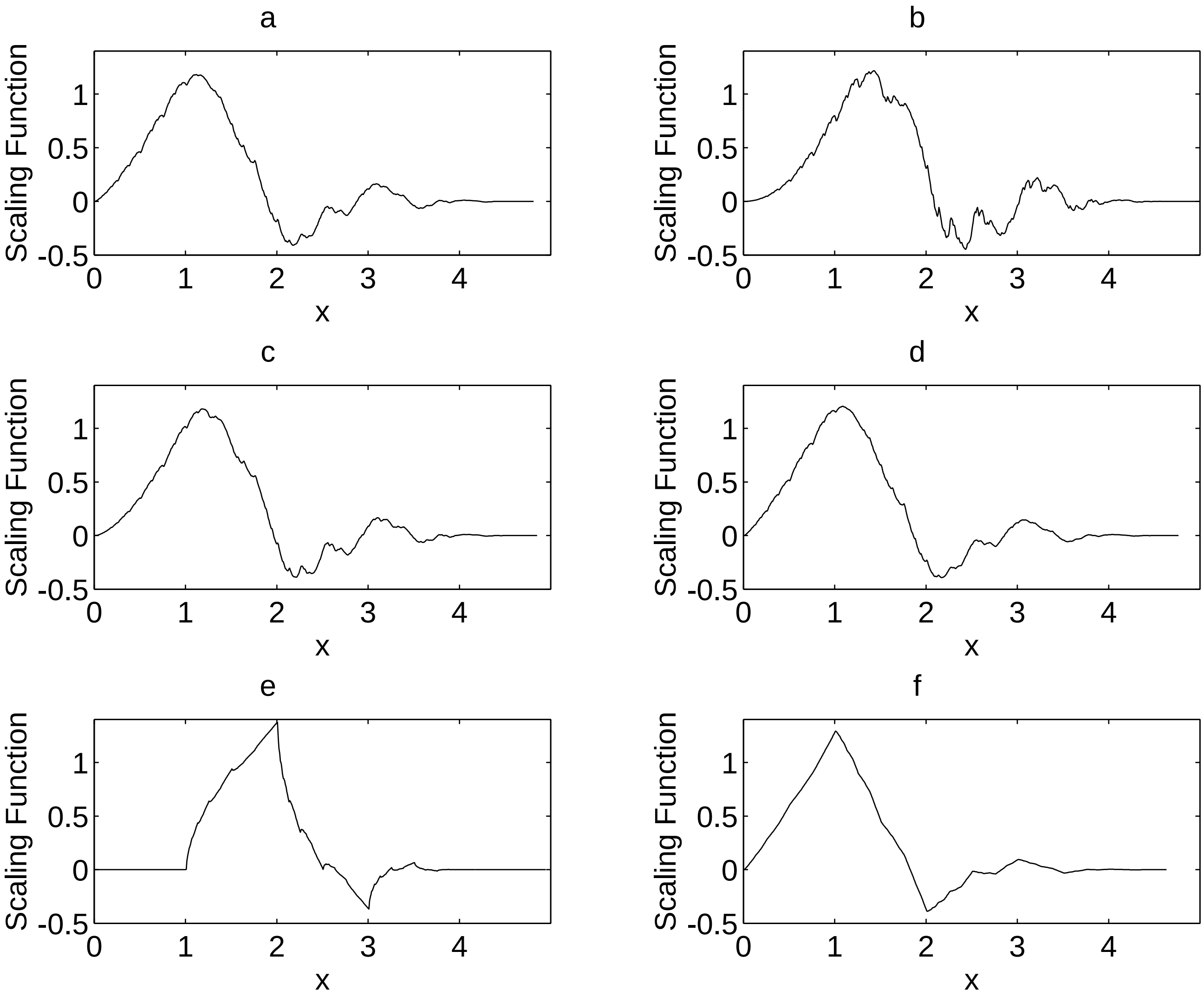

Figure 2.1: The wavelet scaling functions listed and defined in 4.2. Such functions are used as basis functions in the Galerkin-wavelet technique described in this paper. The figures were generated using the successive approximation technique of wavelet scaling functions described in [6]. a. W1. b. W2. c. W3. d. W4. e. D4. f. D6.

problem. We have used closed form expressions for both the analytical and numerical group velocity for acoustic, elastic, and viscoelastic wave propagation to design various cost functions on the form,

$$\sqrt{\int_{PassBand} (F(\omega,\lambda) - \hat{F}(\omega,\lambda,\alpha,\beta))^2 d\omega d\lambda} \tag{4.1}$$

where ω is frequency, λ Courant number, $F(\omega,\lambda)$ is the analytical group velocity, and $\hat{F}(\omega,\lambda,\alpha,\beta)$ is the numerical group velocity. The pass band can be over frequency and/or Courant number. We used the Simplex algorithm in Matlab for the optimization. The method was found to be robust and we have generally been able to find global minima efficiently.

There only exists one 2-point wavelet (the Haar function). The connection coefficients corresponding to this are identical to the $O(h^2)$ centered finite-difference stencil. We have one degree of freedom left when choosing a 4-point wavelet. The Daubechies wavelet (D4) yields connection coefficients equivalent to the $O(h^4)$ centered finite-difference stencil. We have found this choice to be optimal in quite a broad sense. The 6-point wavelets have two degrees of freedom. The traditional centered finite-difference stencil of $O(h^8)$ does not correspond to any wavelet. Similarly to the connection coefficients for the D6 wavelet it yields a scheme with good properties over broad frequency and Courant number intervals, compared to many other choices of stencils. This is something that has been observed in digital signal processing applications of filter banks [16]. Since only limited ranges of Courant numbers, frequencies, etc. occur in wave propagation problems, the Galerkin-Wavelet formulation allows us to design finite-difference stencils with more accurate dispersion than traditional finite-difference schemes within frequency bands of interest. Our technique produced stencils with much better properties than the traditional finite-difference methods both for narrow frequency banded signals (typically less than one octave) over broad Courant number ranges, and also when optimizing broad banded signals (several octaves) for certain velocities and Courant numbers. We will illustrate this through two examples.

4.1 Example 1: Model with small velocity variations

Suppose that we want to simulate wave propagation through a medium with small velocity variations. We wish to perform the simulation using as large Courant number and spatial step size as possible. The highest possible Courant number is given by the stability criterion. Let us assume that we have a stability limit of 0.5. The stability limit generally increases when using a wavelet designed stencil compared to a centered finite-difference scheme. Suppose that we can tolerate say 4 % error for the numerical group velocity, that is a phase shift of 4 wavelengths after a path of propagation of 100 wavelengths compared to the analytical solution. Thus we want to find a wavelet that yields less than 4 % error at minimum grid-points/wavelength ratios. In Figure 4.1 we compare the group velocity dispersion curves for the centered finite-difference scheme to the optimal 6-point wavelet basis. For the centered finite-difference scheme we have 4 % error at 11.5 grid-points/wavelength. For the optimal wavelet choice this occurs at 4.5 grid-points/wavelength. This may lead to considerable savings of computations, especially in 3-D simulations.

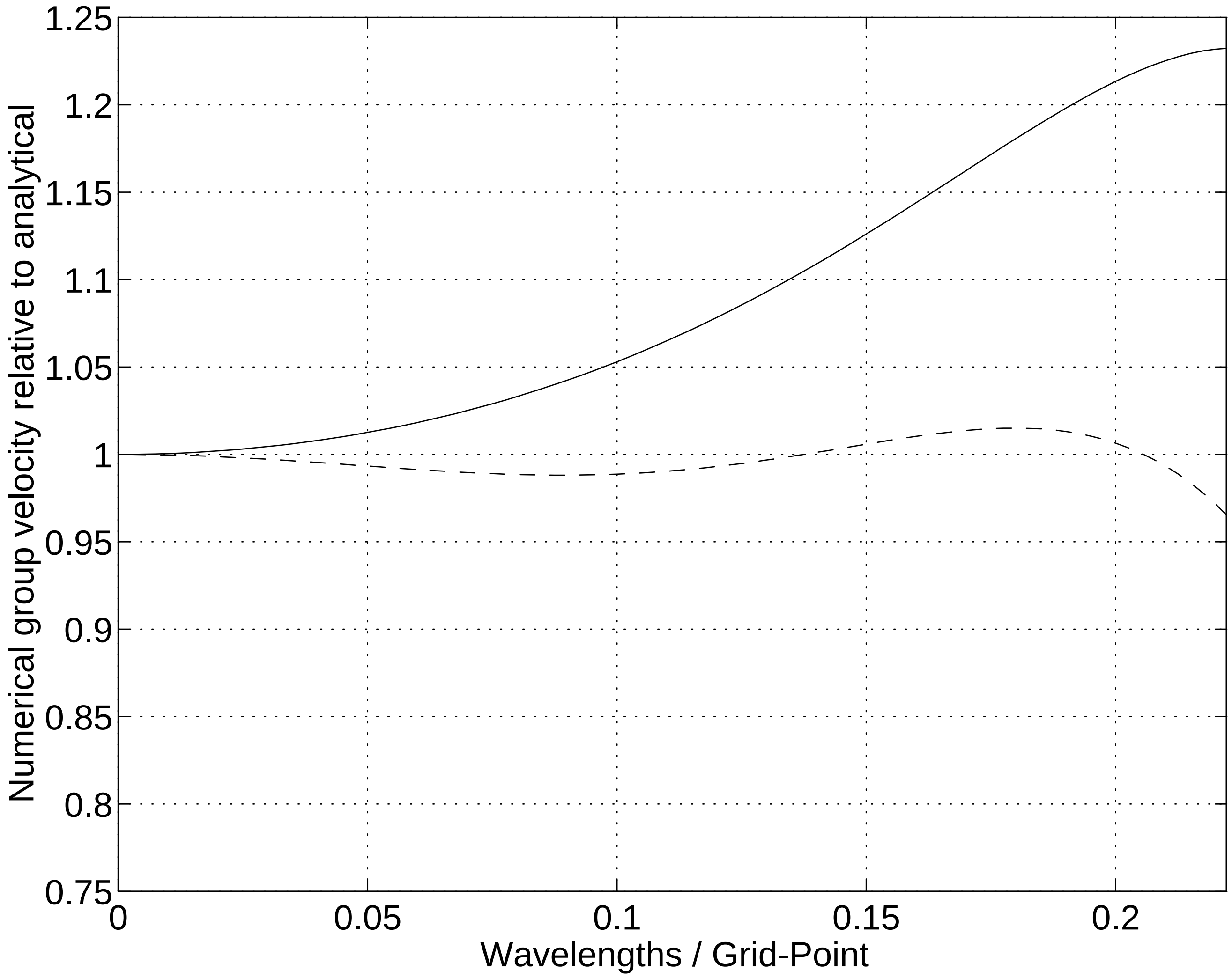

Figure 4.1: A comparison of group velocity dispersion for the centered finite-difference scheme (solid) to the wavelet designed in Example 1 (dashed). The wavelet was designed to yield minimum numerical group velocity dispersion (as close to 1 on the abscissa as possible) up to 4.5 gridpoints/wavelength at a Courant number of 0.5. The optimal wavelet yields less than 4 % dispersion below 4.5 gridpoints/wavelength. It has α=1.0369 and β=-0.7202.

	v_p (km/s)	v_s (km/s)
Basement	2.6	1.45
Sediments	1.6	1.0
Water	1.54	-

Table 4.1: The model used as a test problem. The material velocities are typical for shallow marine environments.

Wavelet	α	β
W1	1.1858	-0.749
W2	0.9725	-0.5433
W3	1.1410	-0.6502
W4	1.2296	-0.7751
D4	1.0472	0
D6	1.3598	-.7821

Table 4.2: W1 through W4 are wavelets optimized in the least square sense using the material properties listed in 4.1, a time step of 0.1069 ms, a spatial step of 0.56 m, and a frequency band between 50 and 300 Hz. For comparison we have included the 4- and 6-point Daubechies wavelets (D4 and D6). W1. Optimized using a cost function based on three different velocities; namely, c=2.6 km/s, c=1.5 km/s, and c=1.0 km/s, with c=1.5 km/s weighted a factor of five in the cost function to obtain an all-around basis with emphasis on c=1.5 km/s since the P-wave in the water and the sediments, the S-wave in the basement, and surface waves (Stonley waves) in the top of the basement layer approximately all propagate with a similar velocity. W2. Optimized for a velocity of 2.6 km/s. W3. Optimized for a velocity of 1.5 km/s. W4. Optimized for a velocity of 1.0 km/s.

4.2 Example 2: Model with large velocity variations

We chose a typical shallow seismic model as test problem. The velocities of the media are listed in 4.1. We assumed a source bandwidth between 50 and 300 Hz, which is typical for sources such as water- and air-guns. A minimum number of grid-points per wavelength of 6 in the slowest medium with S-wave velocity of 1.0 km/s was chosen. It proves to be impossible to find wavelet bases that yield sufficiently small errors in group velocity for much lower grid-points/wavelength ratios than 6, in most practical problems. To maintain 6 grid-points per wavelength at 300 Hz at a velocity of 1.0 km/s, a step size of 0.56 m is required. The maximum velocity is 2.6 km/s. A Courant number of 0.5 at this velocity yields a time-step of 0.1069 ms.

Based on various criteria we designed the optimal stencils of nine grid-points width listed in 4.2. We refer to these wavelets as W1, W2, W3, and W4. The wavelets were obtained through optimization of equation (4.1) over 50-300 Hz at different constant Courant numbers (or velocities since Δt and Δx already are fixed) for W2 through W4 and a discrete range of Courant numbers (or velocities) for W1. We have plotted the group velocity over the frequency interval of interest

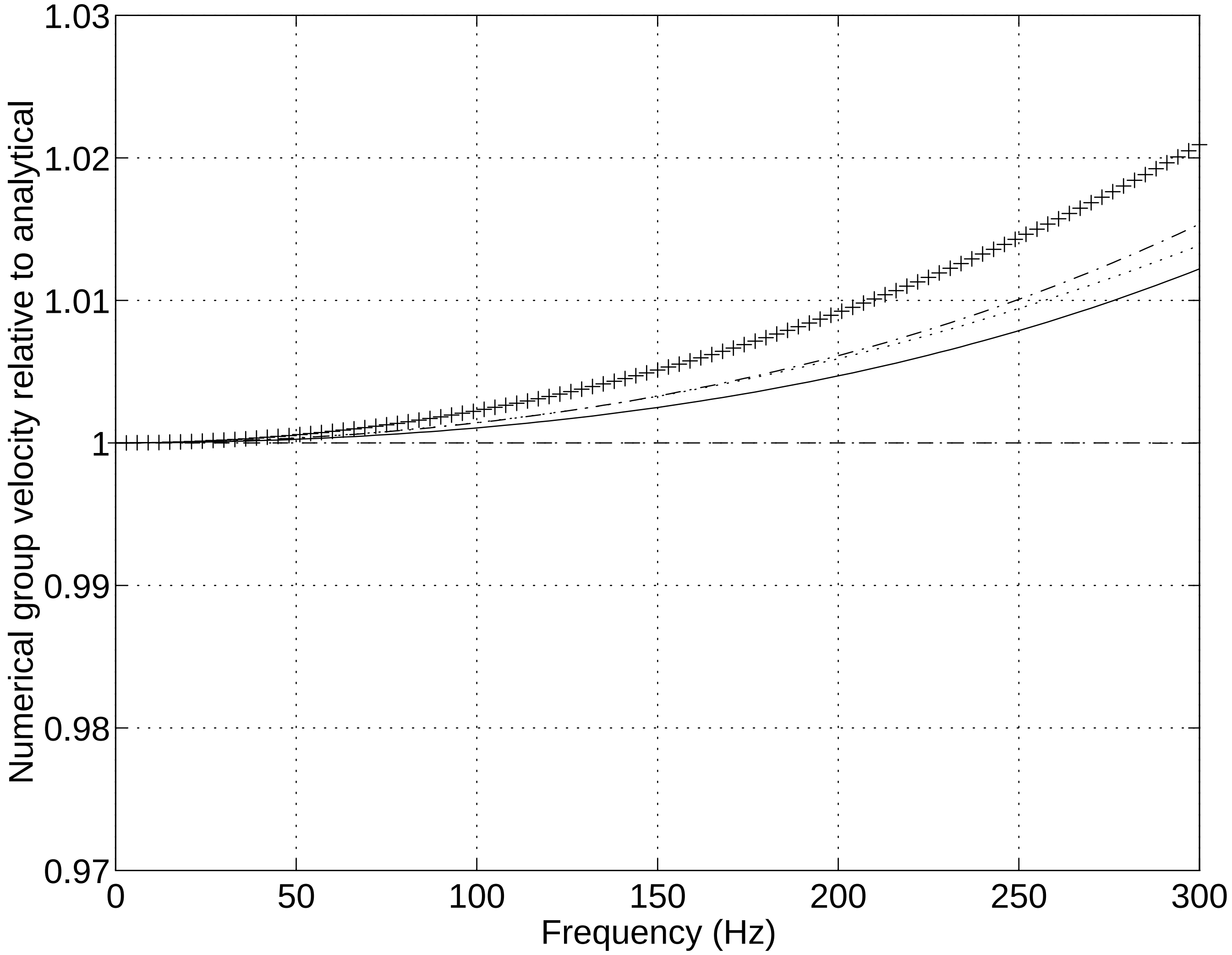

Figure 4.2: A comparison of group velocity dispersion for the centered finite-difference scheme (marked by +) to the wavelets designed in Example 2 (see 4.2). Δt=0.1069 ms. Δx=0.56 m. c=2.6 km/s. This yields a Courant number of 0.5. An optimal dispersion relation would be constant and equal to one over the entire interval. Solid: W1, Dashed: W2, Dotted: W3, Dash-Dotted: W4, Plus: centered finite-difference scheme.

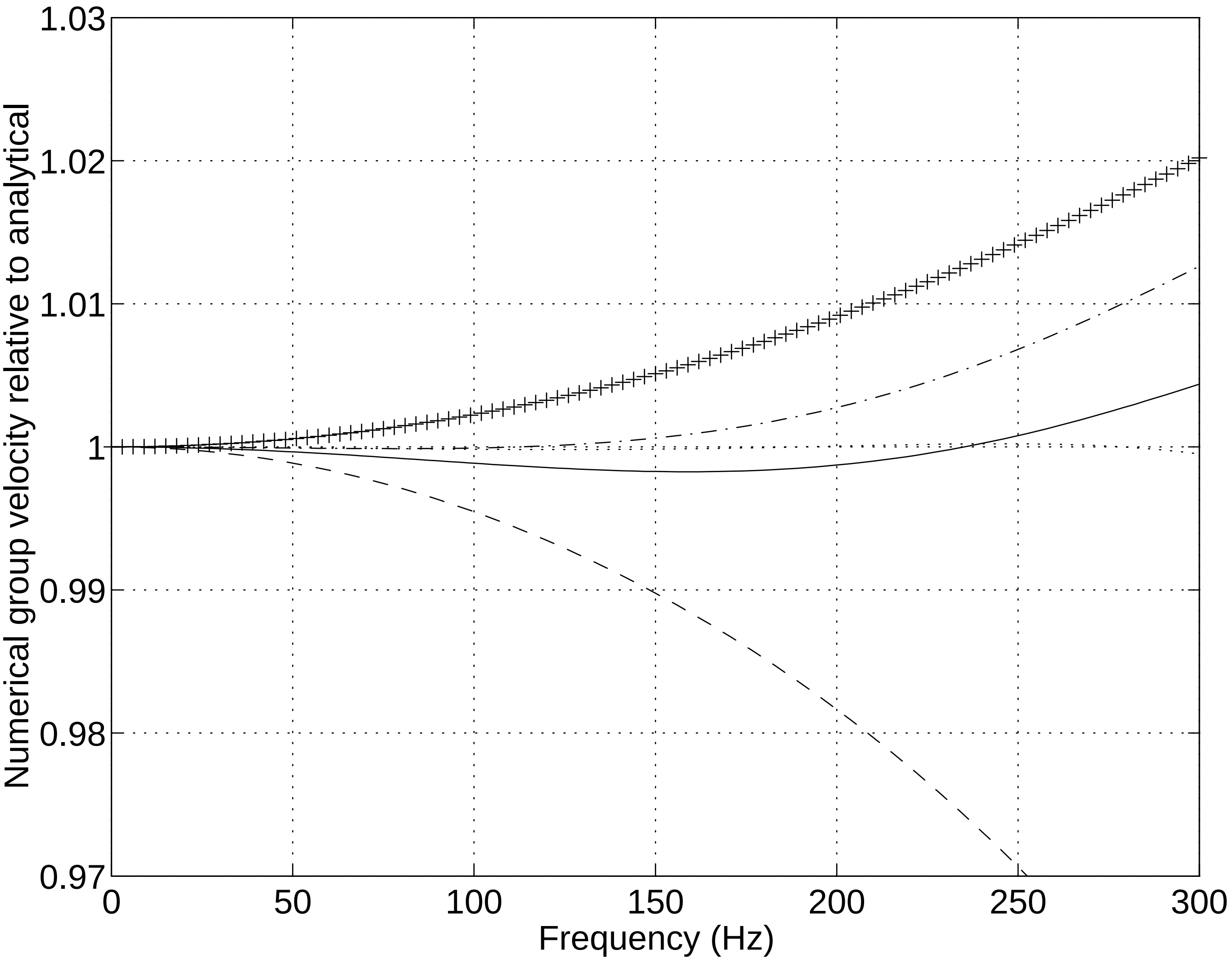

Figure 4.3: A comparison of group velocity dispersion for the centered finite-difference scheme (marked by +) to the wavelets designed in Example 2 (See 4.2). Δt=0.1069 ms. Δx=0.56 m. c=1.5 km/s. This yields a Courant number of 0.29. An optimal dispersion relation would be constant and equal to one over the entire interval. Solid: W1, Dashed: W2, Dotted: W3, Dash-Dotted: W4, Plus: centered finite-difference scheme.

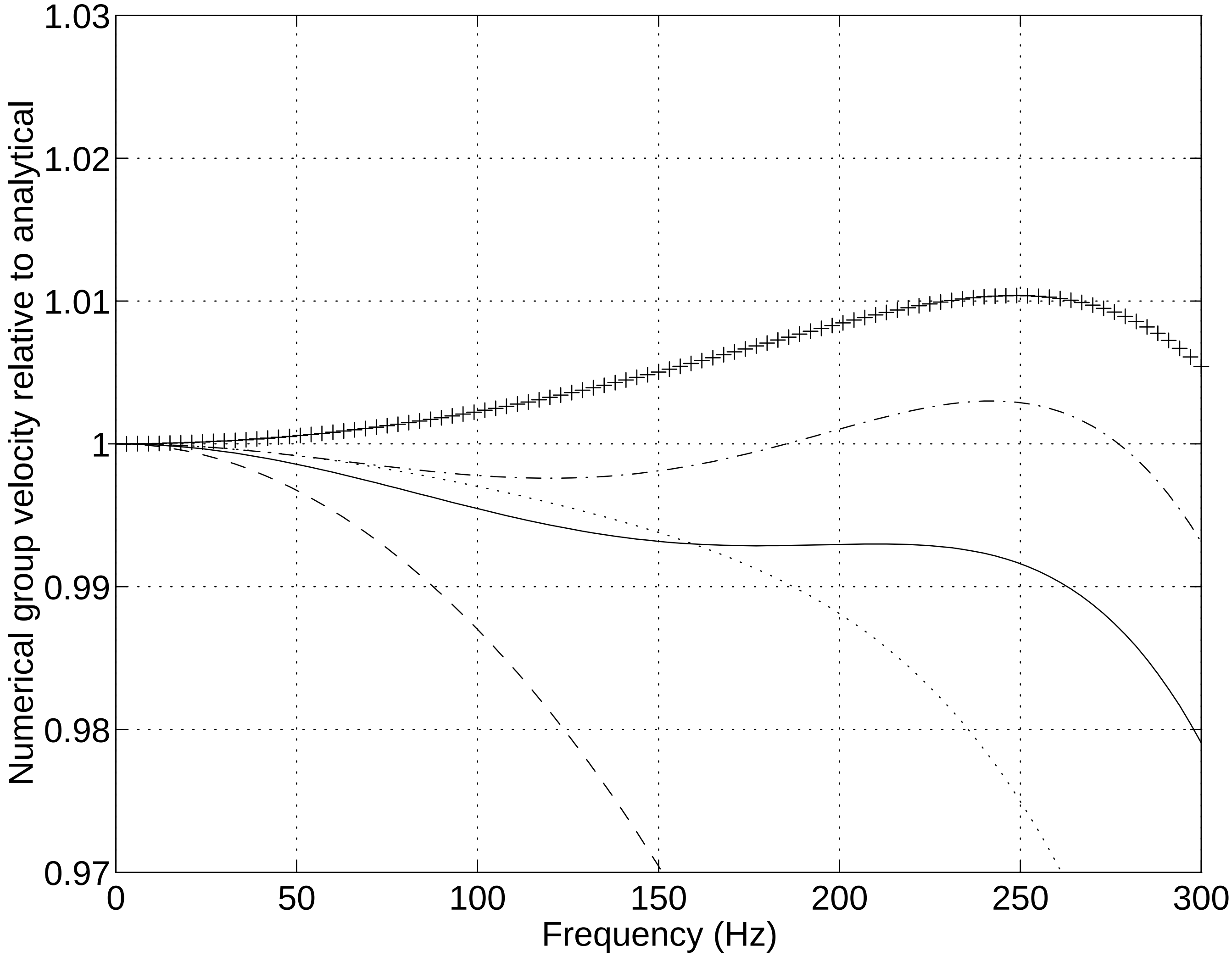

Figure 4.4: A comparison of group velocity dispersion for the centered finite-difference scheme (marked by +) to the wavelets designed in Example 2 (See 4.2). Δt=0.1069 ms. Δx=0.56 m. c=1.0 km/s. This yields a Courant number of 0.2. An optimal dispersion relation would be constant and equal to one over the entire interval. Solid: W1, Dashed: W2, Dotted: W3, Dash-Dotted: W4, Plus: centered finite-difference scheme.

for the different velocities in Figures 4.2-4.4. The centered finite-difference stencil of eighth order generally has a decent behavior over wide ranges of Courant numbers and frequencies. For a particular problem this is a waste since only limited ranges of Courant number and frequencies occur. Compared to the other stencils, the centered finite-difference scheme improves at lower Courant numbers.

Both the W1 and W4 wavelet have errors in the group velocity, which are less than 2 % over the full frequency range in all media. This is superior to the centered finite-difference scheme. If we consider 1.5 km/s to be the most important velocity to model well, then the W3 scheme should be optimal with an almost perfect group velocity. The W2 scheme similarly displays an almost perfect group velocity at 2.6 km/s.

5 Discussion and conclusions

In this paper we have outlined a technique to design optimal finite-difference stencils using a Galerkin-wavelet technique. The Galerkin-wavelet formulation reduces to explicit finite-difference modeling. We have shown that for fixed Courant number and frequency ranges optimal choice of wavelet bases gives an alternate choice to the traditional centered finite-difference stencils obtained from Taylor expansions with superior dispersion characteristics. Use of these optimal wavelet-based schemes results in considerable gain in accuracy as well as large computational savings. The wavelet parameter optimization technique generally converges to global minima of the RMS group velocity error criterion.

The Galerkin-wavelet formulation also opens up a variety of new possibilities. For example Galerkin finite element theory provides a standard means for stable inclusion of boundary conditions. Both essential and natural boundary conditions will alter the wavelet-Galerkin stencils, in some cases resulting in implicit time-stepping for near boundary nodes. Since only a small fraction of the nodes are affected, the mass matrix can be rediagonalized in a preprocessing step at reasonable cost. Alternative by wavelet families adapted to essential boundary conditions might be used [2, 11].

Recently work by Gopinath and Burrus [8] has shown that spatially varying families of orthonormal wavelets exist; in effect the parameters α, β become functions of spatial position. This discovery suggests wavelet-Galerkin schemes in which the dispersion characteristics are optimized locally, rather than globally. Such schemes might be particularly efficient for simulation of waves in heterogeneous media. They would retain explicit time stepping, natural boundary conditions, and logically uniform, rectangular grid of the schemes presented above, while enjoying optimal local dispersion.

6 Acknowledgments

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