Viscoelastic Finite Elements

Kirk Blazek

Department of Computational and Applied Mathematics
Rice University

January 26, 2007

The Lions Method for Solving Hyperbolic PDEs

Christian Stolk, in his 2000 PhD thesis updated the classic method of Lions for solving hyperbolic problems.
This method takes a second-order hyperbolic equation, such as the elastic wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \left(C \frac{1}{2} \left(\nabla u + \nabla u^T \right) \right),$$

and rewrites it in an operator form:

$$(A(t)u'(t))' + D(t)u(t) + B(t)u(t) = f(t).$$

The Lions Method and Finite Elements

Existence and uniqueness for the problem

$$(A(t)u'(t))' + D(t)u(t) + B(t)u(t) = f(t).$$

is then proven by approximating u in the spacial direction

$$u_m(t) = \sum_{k=1}^{m} g_{km}(t)w_k.$$

These approximations solve ordinary differential equations, and they converge thanks to energy inequalities.
This approximation and convergence is then a theoretical justification of the method of finite elements.

- Stolk's results are currently unpublished.

Viscoelastic Finite Elements

Our question is, can we apply this same technique to the viscoelastic wave equation

$$
\begin{aligned}
\rho \frac{\partial v_i}{\partial t} &= \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \\
\frac{\partial \sigma_{kl}}{\partial t} &= \sum_{i,j} C_{ijkl} *_t \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),
\end{aligned}
$$

and if so, what additional information does this give us?

The Lions Method for First-Order Systems

The answer is yes, we can apply the Lions method to viscoelasticity. The resulting existence and uniqueness proof then gives us a few additional facts.

The Lions Method for First-Order Systems

The answer is yes, we can apply the Lions method to viscoelasticity. The resulting existence and uniqueness proof then gives us a few additional facts.

- Justification of the finite element method for general hyperbolic integro-differential equations, not just first-order hyperbolic equations.

The Lions Method for First-Order Systems

The answer is yes, we can apply the Lions method to viscoelasticity. The resulting existence and uniqueness proof then gives us a few additional facts.

- Justification of the finite element method for general hyperbolic integro-differential equations, not just first-order hyperbolic equations.
- The energy inequalities used in the proof also give continuity results for the problem, not just for the initial conditions and forcing functions, but continuity of the solution with respect to the coefficients.

The Lions Method for First-Order Systems

The answer is yes, we can apply the Lions method to viscoelasticity. The resulting existence and uniqueness proof then gives us a few additional facts.

- Justification of the finite element method for general hyperbolic integro-differential equations, not just first-order hyperbolic equations.
- The energy inequalities used in the proof also give continuity results for the problem, not just for the initial conditions and forcing functions, but continuity of the solution with respect to the coefficients.
- The solution of the first-order differential equation maintains the hyperbolicity seen in the second-order equation, but it manifests itself differently.

Operator Form of the Equation

We rewrite the viscoelastic equation in operator form,

$$A(t)u'(t) + D(t)u(t) + B(t)u(t) + R[u](t) = f(t),$$

where D takes the place of the spacial derivatives, B is a general lower-order term, and R is the integral term representing viscoelasticity defined by

$$R[u](t) = \int_0^T Q(s,t)u(s)\,ds,$$

Theorem

Suppose $u_0 \in H$, $f \in L^2([0,T],H)$, the the differential equation has a unique solution $u \in L^2([0,T],H)$ which depends continuously on u_0 and f.

Sketch of proof

For simplicity we assume that H is separable. Let $\{w_k\}_{k=1}^{\infty} \subset V$ form a basis for H in the sense that finite linear combinations of w_k's are dense in H. Define the functions

$$u_m(t) = \sum_{k=1}^{m} g_{km}(t) w_k,$$

where the g_{km}'s are determined by the differential equation

$$\begin{array}{rcl} \langle u_m'(t), A(t)w_l\rangle - \langle u_m(t), D(t)w_l\rangle & & \\ +\langle u_m(t), B(t)w_l\rangle + \langle u_m(t), R^*[w_l](t)\rangle & = & \langle f(t), w_l\rangle, \\ u_m(0) & = & \xi_{km}, \end{array}$$

for $1 \leq l \leq m$.

Continuity

Theorem

Consider the equations

$$\begin{aligned}
\widetilde{A}\tilde{u}'(t) + \widetilde{D}(t)\tilde{u}(t) + \widetilde{B}(t)\tilde{u}(t) + \widetilde{R}[\tilde{u}](t) &= f(t), \\
Au'(t) + D(t)u(t) + B(t)u(t) + R[u](t) &= f(t).
\end{aligned}$$

If the coefficients are close in norm independent of t, A is positive definite, and R and B are positive semi-definite, then $\tilde{u}$ is close to u in $L^2([0,T],H)$.

Theorem

Differentiation with respect to the coefficients of the differential equation result in solutions with one less order of regularity in the spacial direction.

Hyperbolicity

Returning to the elastic wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \left(C \frac{1}{2} \left(\nabla u + \nabla u^T \right) \right),$$

if we have a solution $u \in C^1([0,T], H^1(\mathbb{R}^n))$, then $\partial u / \partial t \in C([0,T], L^2(\mathbb{R}^n))$.
For the first-order system

$$\begin{aligned} \rho \frac{\partial v_i}{\partial t} &= \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \\ \frac{\partial \sigma_{kl}}{\partial t} &= \sum_{i,j} C_{ijkl} *_t \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \end{aligned}$$

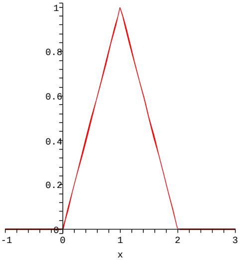
we can no longer differentiate in time. So what is the equivalent property for this system?

Hyperbolicity - A One-D Example

As an example, consider the basic one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

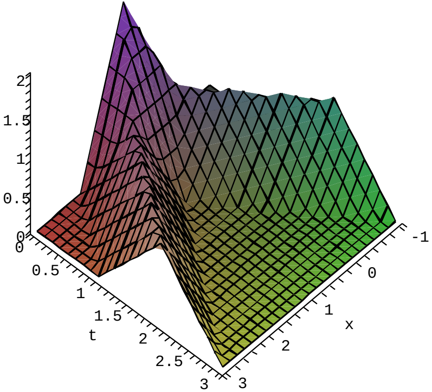
with initial condition $u(x, 0)$ of the form



and $\partial u / \partial t(x, 0) = 0$.

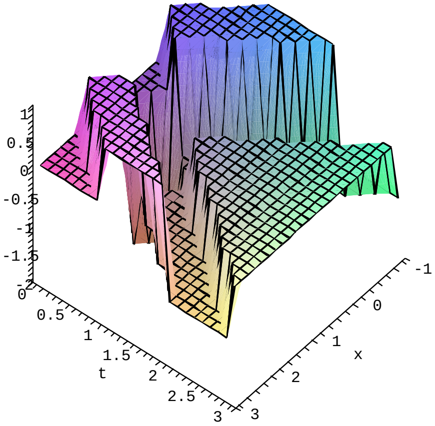
A One-D Example

Solving this equation gives the solution



A One-D Example

If you take a derivative in time, you get a solution that looks like



which is also less continuous in the x direction.

Hyperbolicity From the First-Order System

This is true in general. For a hyperbolic equation, if you differentiate in time, you lose one order of regularity in space.

- The question: How can we see this phenomenon in the first order system?
- The answer: if we smooth the solution in time, it should also become smoother in space.

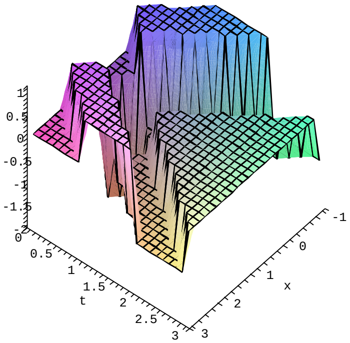
In other words, if we have a solution $u \in C([0,T], L^2(\mathbb{R}^n))$, then

$$\begin{aligned} \eta * u(x,t) &= \int_{-\infty}^{\infty} u(x,\tau)\eta(t-\tau)\, d\tau \\ &\in C^\infty([0,T], H^1(\mathbb{R}^n)), \end{aligned}$$

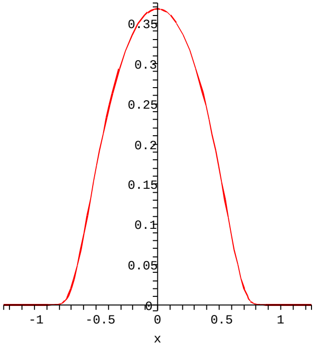
for any $\eta \in C_c^\infty(0,T)$.
Recall that a function can be "made smoother" by taking the convolution with a smooth function. For instance, if $f \in L^2(\mathbb{R})$ and $g \in H^1(\mathbb{R})$, then $f * g \in H^1(\mathbb{R})$ since $(f*g)' = f * g'$.

Smoothing Example

Returning to the prior example. Let's pretend we have a solution to the wave equation and a smoothing function.



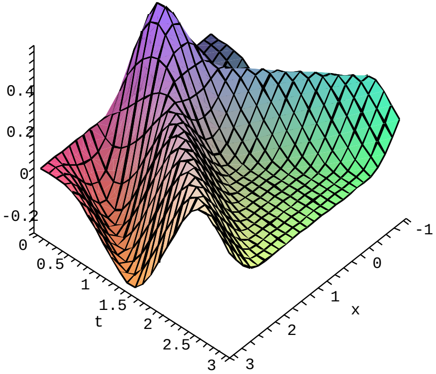
u



η

Smoothing Example

Taking the time convolution of η with u gives



which is nice and smooth.