Effective Waveform Inversion via nonlinear DSO

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The Rice Inversion Project

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Introduction

Focus:

- review Dong's MS work on "Nonlinear DSO for Plane-waves in Layered Media"
- discuss an important revision to the above Nonlinear DS formulation

Remark: the nonlinear DS approach in Dong's thesis is an application of the Extended Modeling Concept (Symes,2008), which permits a unified framework for WI and MVA, and may lead to effective WI.

Outline

Waveform Inversion (WI)

The usual set-up:

- $\mathcal{M}$: Model Space (possible models of earth structure)
- $\mathcal{D}$: Data Space
- $\mathcal{F} : \mathcal{M} \to \mathcal{D}$: modeling operator (Forward Map)

WI problem:

given $d \in \mathcal{D}$, find $m \in \mathcal{M}$ such that $\mathcal{F}[m] \approx d$

often in the form of **Output Least Squares Inversion**:

$$\min_{m \in \mathcal{M}} J_{OLS} := \frac{1}{2} \|\mathcal{F}[m] - d\|_{\mathcal{D}}^2 + \mathcal{R}(m)$$

Overview of Output Least Squares Inversion

Pros:

- take into account any physics
- reconstruct detailed models of subsurface structure

Approaches:

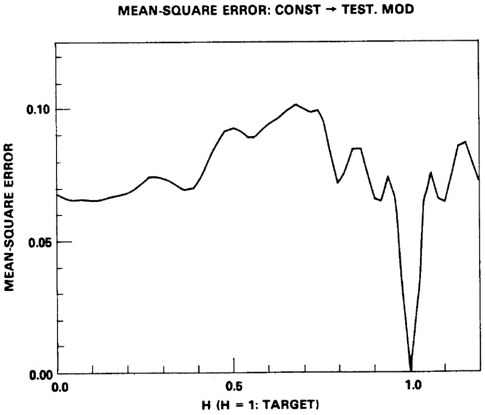
- Global methods (infeasible)
  simulated annealing, genetic method, etc.
- Local methods (Gradient-related approaches)

Problem Size $\Rightarrow$ Gradient-related approaches

But:
J_{OLS} has lots of useless local minima for typical set-up of exploration seismology
$\Rightarrow$ least squares inversion with any Newton-related approach doesn't work

OLS Inversion: Fundamental Impediment

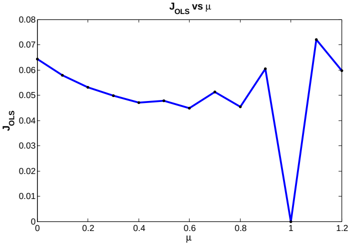
J_{OLS} possess lots of useless local minima (for typical data)
$\Longrightarrow$ Newton-like iteration stagnates at some local minimum far away from the global one



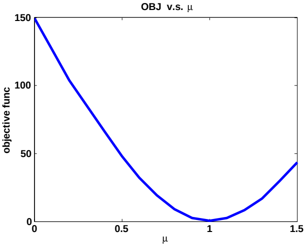
(Symes & Carazzone 92)

Why the proposed strategy matters?

How to turn lots of this ... into this?



$\stackrel{?}{\Longrightarrow}$

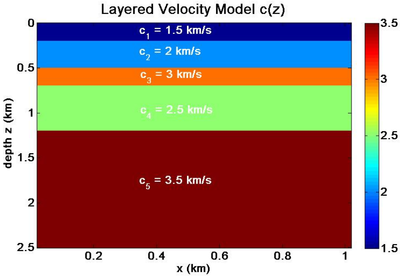


Outline

Nonlinear DSO for plane-waves in layered media

1. Problem Set-up
2. One Observation and nonlinear DS Strategy
3. Scan Tests

Constant Density Layered Acoustic Model



z: depth

$c(z)$: acoustic velocity

$u(x, z, t)$: wave-field potential

$\omega(t)$: source time function

ξ: slowness

$\mathcal{M} := \{c(z) : \ldots\}$

Wave Equation for $u(x, z, t)$:

$$\left(\frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \right) u(x, z, t) = \omega(t) \delta(x, z)$$

Introduce Slant Stacked field (Radon Transform)

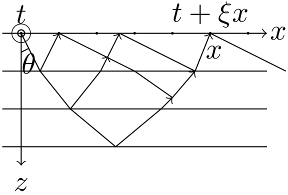
$$U(\xi, z, t) = \int dx \, u(x, z, t + \xi x)$$

Plane-wave Decomposition

Radon transform:

$$U(\xi, z, t) = \int dx\, u(x, z, t + \xi x)$$

$$\xi = \frac{sin(\theta)}{c(z)}$$



$\Longrightarrow$ a set of 1D plane wave problems

$$\left(\frac{1}{v^2(\xi, z)}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right) U(\xi, z, t) = \omega(t)\delta(z)$$

$v = \frac{c(z)}{\sqrt{1 - c^2(z)\xi^2}}$ for $|\xi \cdot c(z)| < 1$ (**vertical velocity**)

Forward Map: $\mathcal{F}_\omega[c] := \frac{\partial}{\partial t} U(\xi, 0, t)$

General Inverse Problem in Extension Form

Inverse Problem: given data $d \in \mathcal{D}$, find c such that $\mathcal{F}_\omega[c] \simeq D$

Recall: for each slowness ξ, $\mathcal{F}_\omega[c](\xi,t) = d(\xi,t)$ poses a 1D problem

$$\mathcal{F}_\omega[c](\xi,t) = d(\xi,t) \xrightarrow{\text{1D OLS inversion}} v(\xi,z) \xrightarrow{c = \frac{v}{\sqrt{1+v^2\,\xi^2}}} \bar{c}(\xi,z)$$

$\bar{c}$ is physically meaningful only if $\frac{\partial \bar{c}}{\partial \xi} = 0$

A new form of the inverse problem

$$\min_{\bar{c}\in\overline{\mathcal{M}}} \quad J_{DS}[\bar{c}] := \frac{1}{2}\left\|\frac{\partial \bar{c}}{\partial \xi}\right\|^2$$

$$s.t. \qquad \left\|\overline{\mathcal{F}}_\omega[\bar{c}] - d\right\|_{\mathcal{D}}^2 \simeq 0$$

$\overline{\mathcal{M}} = \{\bar{c}(\xi,z) : \texttt{positive functions}, \ldots\}$: Extended Model Space

Question: how to navigate through the feasible set

$$S = \left\{\bar{m} \in \overline{\mathcal{M}} : \left\|\overline{\mathcal{F}}_\omega - d\right\|_{\mathcal{D}}^2 \simeq 0\right\} \quad ?$$

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Nonlinear DSO for plane-waves in layered media

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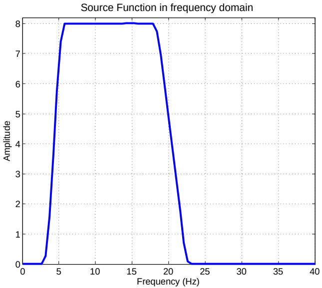
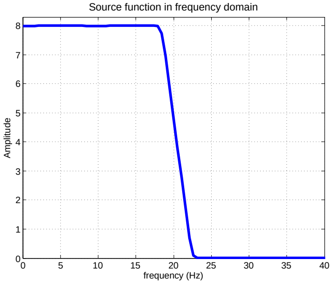
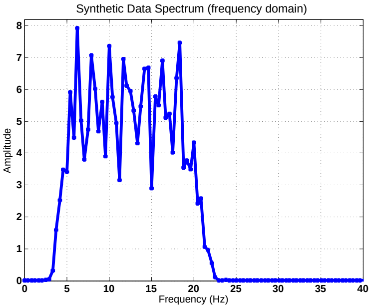
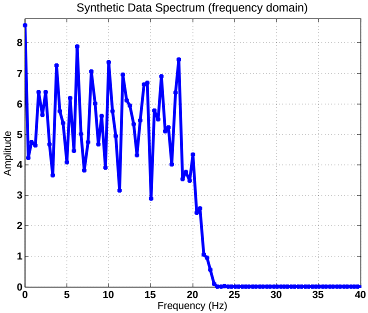
Low-frequencies' Influence in 1D LS Inversion

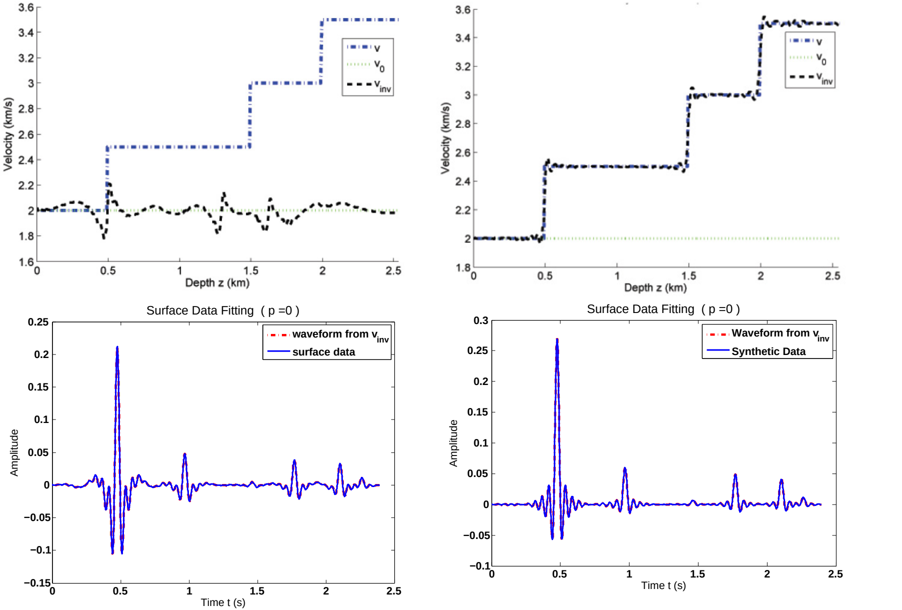




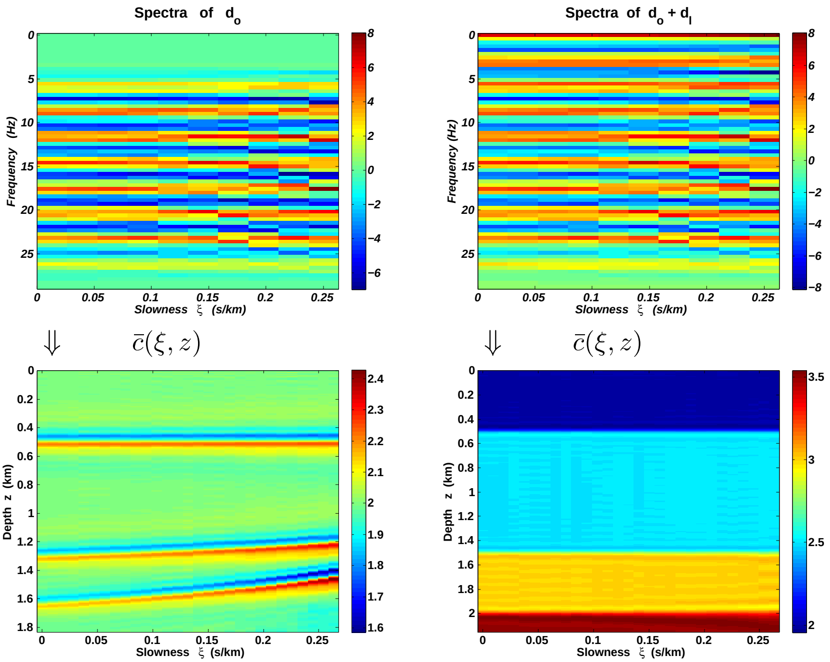




trace $\xi = 0$



Velocity (km/s)
Depth z (km)
v
v0
vinv
Surface Data Fitting (p =0)
waveform from vinv
surface data
Amplitude
Time t (s)
Surface Data Fitting (p =0)
Waveform from vinv
Synthetic Data



Spectra of d_o
Spectra of $d_o + d_l$
Frequency (Hz)
Slowness ξ (s/km)
$\Downarrow$ $\bar{c}(\xi, z)$
$\Downarrow$ $\bar{c}(\xi, z)$
Depth z (km)
Slowness ξ (s/km)

nDS Strategy: Recover missing low-frequency components

Conjecture: suppose source is impulsive with full bandwidth down to 0 Hz, then $\bar{c}$ uniquely determined by data

Then, use low frequency data components, missing from field data, as control parameters, permitting navigation through the feasible set

$$S = \left\{ \bar{m} \in \overline{\mathcal{M}} : \left\| \overline{\mathcal{F}}_\omega - d \right\|_{\mathcal{D}}^2 \simeq 0 \right\}$$

Analogy:

- Low-frequency data components $\leftrightsquigarrow$ Low-frequencies in model
- macromodel in MVA

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Analogy:

- Low-frequency data components $\leftrightsquigarrow$ Low-frequencies in model
- macromodel in MVA

Nonlinear Differential Semblance: Formulation

Use low frequency data components, missing from field data, as control parameters ...

That is, minimizing the following problem to recover missing low-frequency data components

$$\begin{aligned} &\min_{d_l \in \overline{\mathcal{D}_l}} \quad J_{DS}[\bar{c}[d_l]] := \frac{1}{2} \left\| \frac{\partial \bar{c}[d_l]}{\partial \xi} \right\|_W^2 \\ &s.t. \qquad \bar{c}[d_l] = \operatorname{argmin}_{\bar{c} \in \overline{\mathcal{M}}} \left(\frac{1}{2} \left\| \overline{\mathcal{F}}_{\omega+\omega_l}[\bar{c}] - (d_o + d_l) \right\|_{\mathcal{D}}^2 \right) \end{aligned}$$

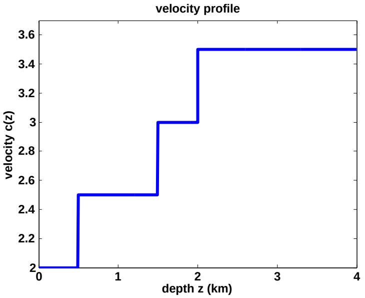
$\mathcal{D}_l$: low-frequency data space, d_l: low-frequency control

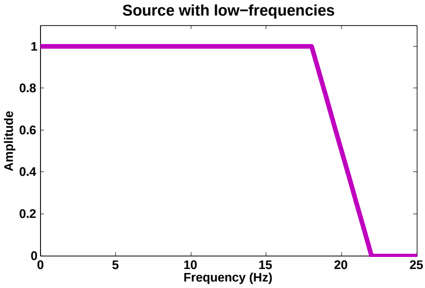
ω_l: fixed low-frequency source components

(s.t. $\omega + \omega_l$ impulsive with full bandwidth down to 0 Hz)

Nonlinear DSO for plane-waves in layered media

Scan Tests: Four-layer Model





⋆Scan J_{DSO} along : $d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad (\mu \in [0, 1.5])$

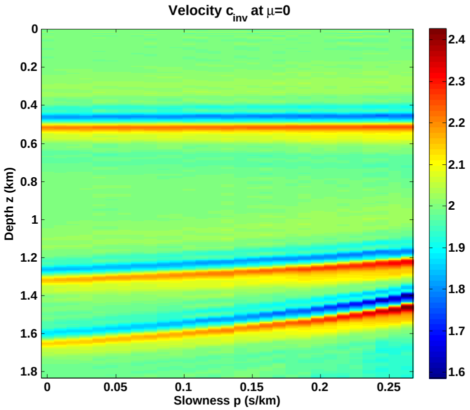
$d_{lpert} = d^*_{high} + d^{hom}_{low}$

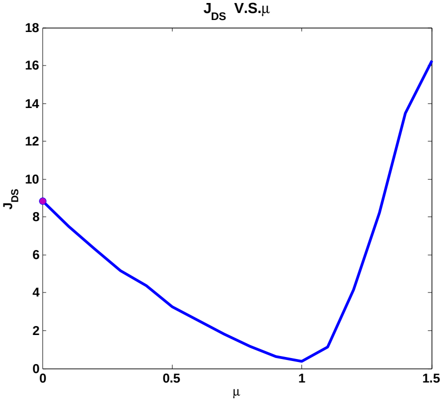
d^{hom} derived from the homogeneous velocity model $c_{hom}(z) \equiv 2$

⋆ Scan J_{OLS} along: $c_\mu(z) = (1-\mu)\, c_{hom} + \mu\, c^*(z) \quad (\mu \in [0, 1.5])$

Scan Tests (Four-layer Model): $\mu = 0$

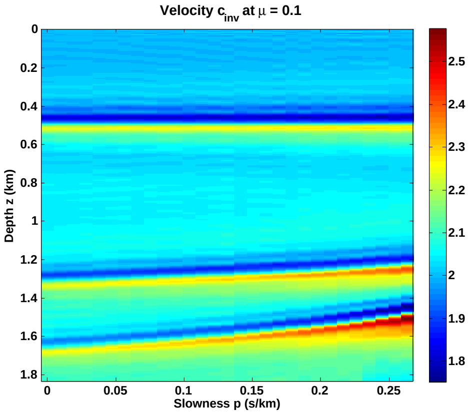
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \quad \mu = 0$$

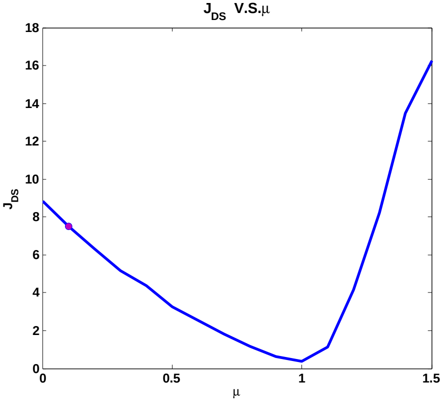




Scan Tests (Four-layer Model): $\mu = .1$

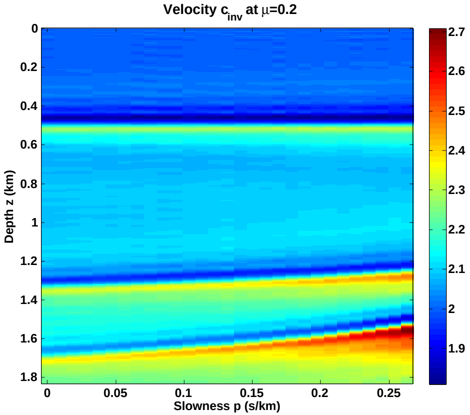
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \quad \mu = 0.1$$

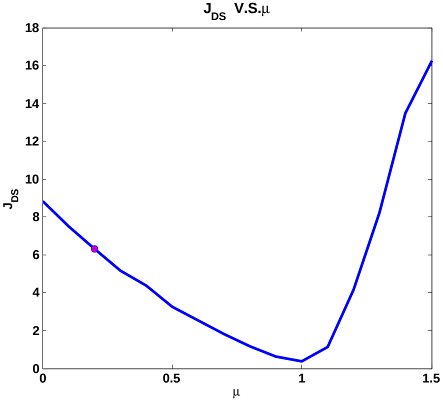




Scan Tests (Four-layer Model): $\mu = .2$

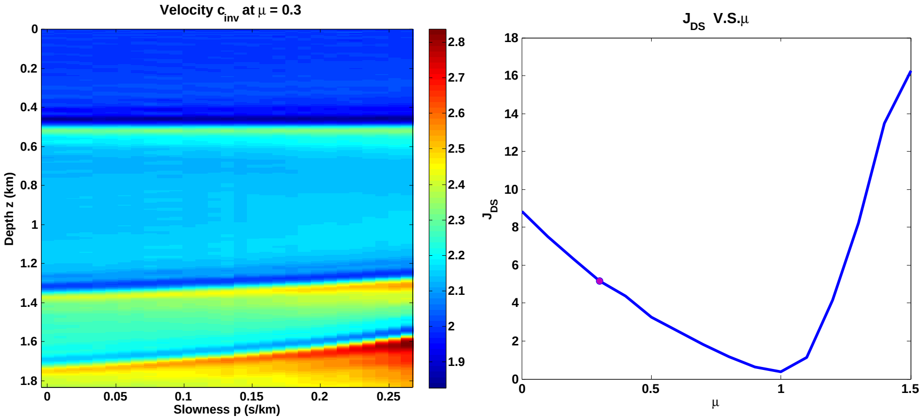
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \;\; \mu = 0.2$$





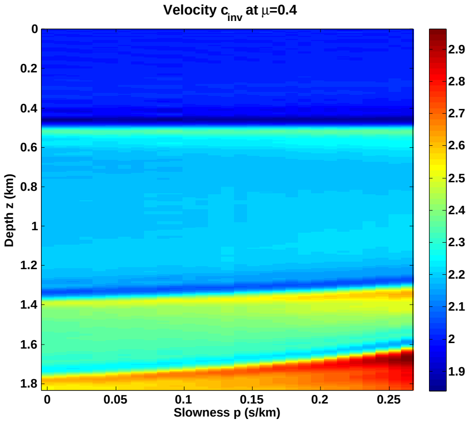
Scan Tests (Four-layer Model): $\mu = .3$

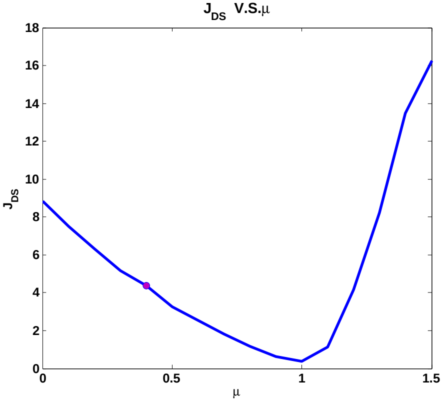
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \quad \mu = 0.3$$



Scan Tests (Four-layer Model): $\mu = .4$

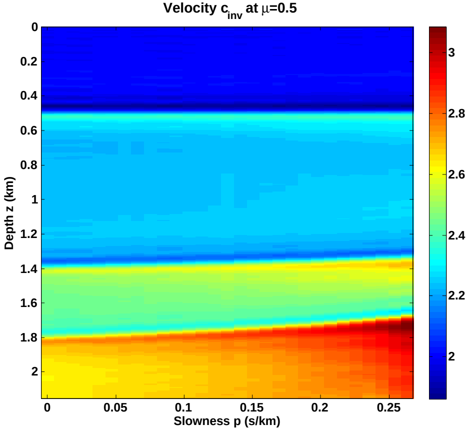
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \quad \mu = 0.4$$

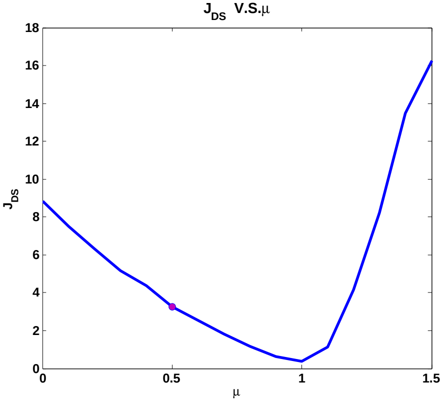




Scan Tests (Four-layer Model): $\mu = .5$

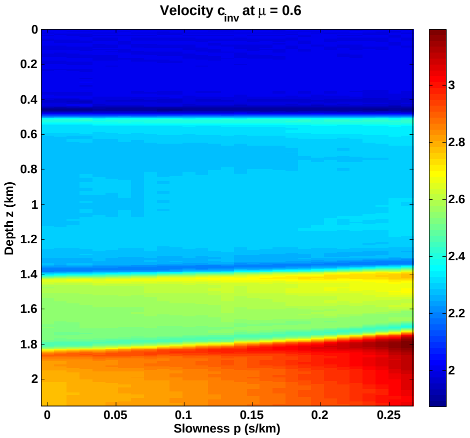
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \quad \mu = 0.5$$

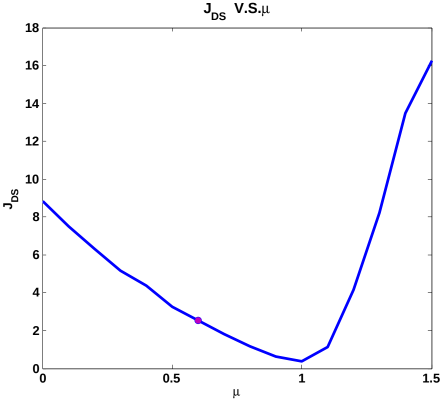




Scan Tests (Four-layer Model): $\mu = .6$

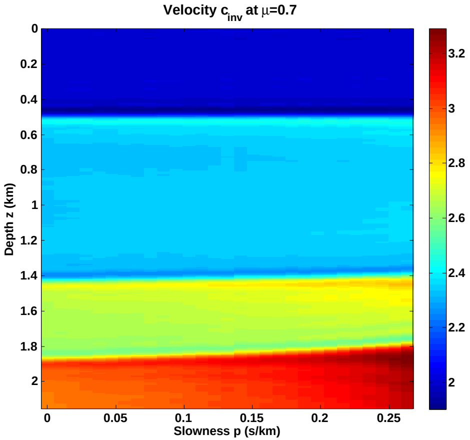
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \;\; \mu = 0.6$$

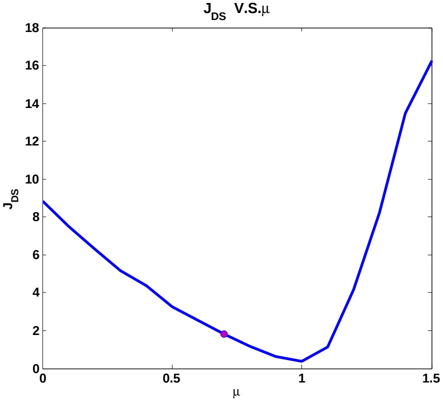




Scan Tests (Four-layer Model): $\mu = .7$

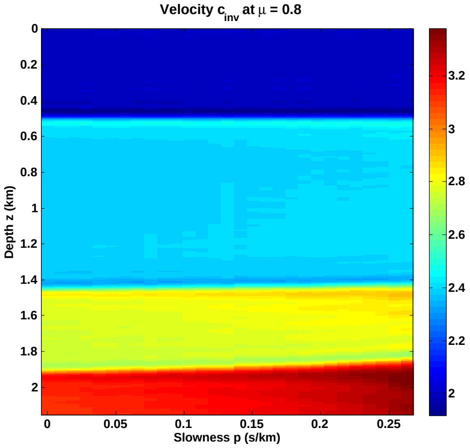
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \quad \mu = 0.7$$

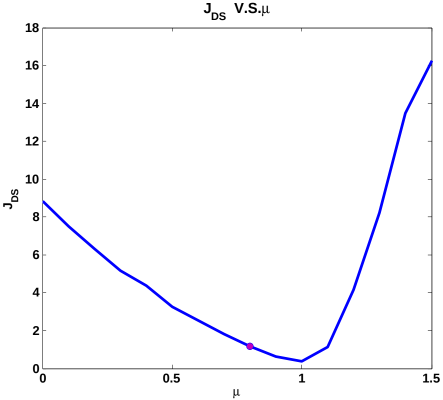




Scan Tests (Four-layer Model): $\mu = .8$

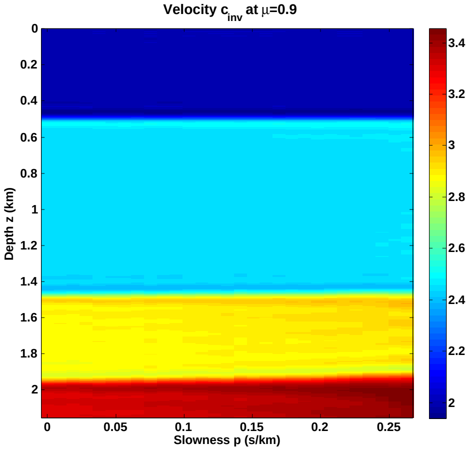
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \;\; \mu = 0.8$$

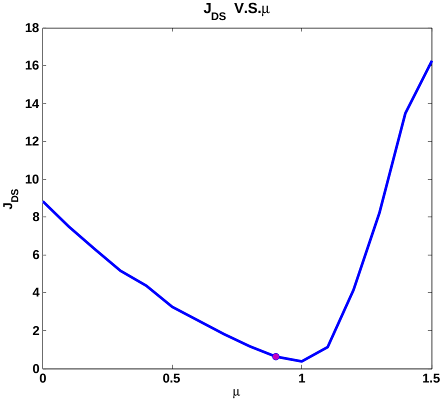




Scan Tests (Four-layer Model): $\mu = .9$

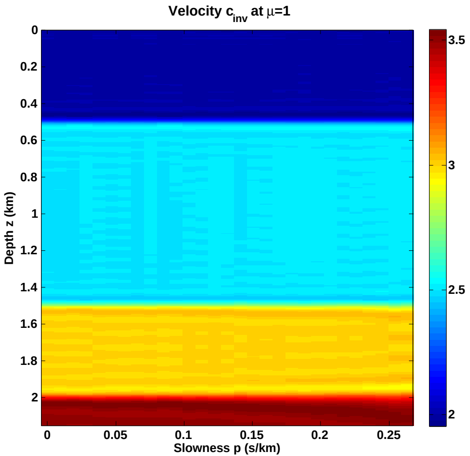
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \quad \mu = 0.9$$

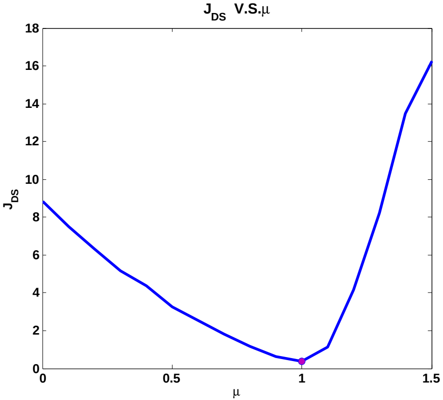




Scan Tests (Four-layer Model): $\mu = 1.0$

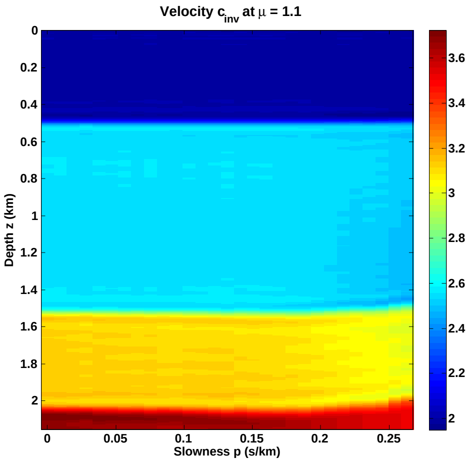
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \;\; \mu = 1.0$$

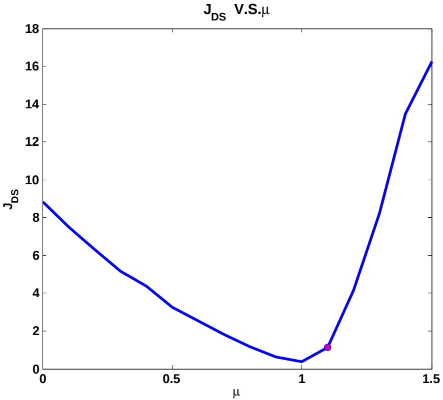




Scan Tests (Four-layer Model): $\mu = 1.1$

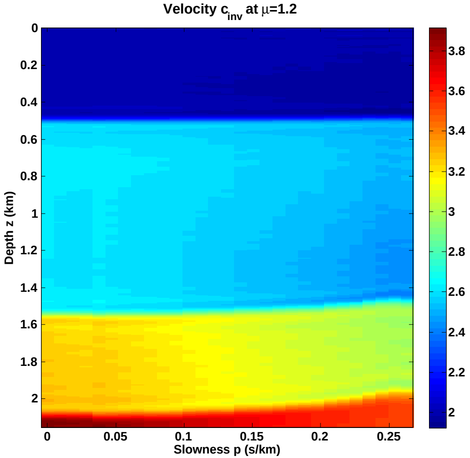
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \;\; \mu = 1.1$$

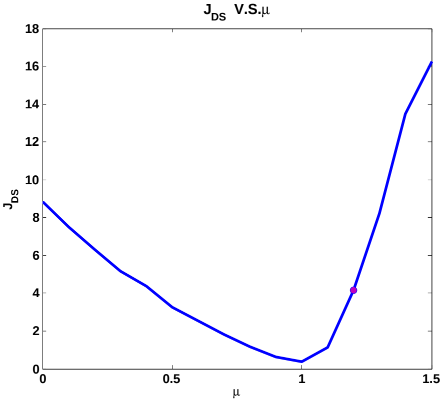




Scan Tests (Four-layer Model): $\mu = 1.2$

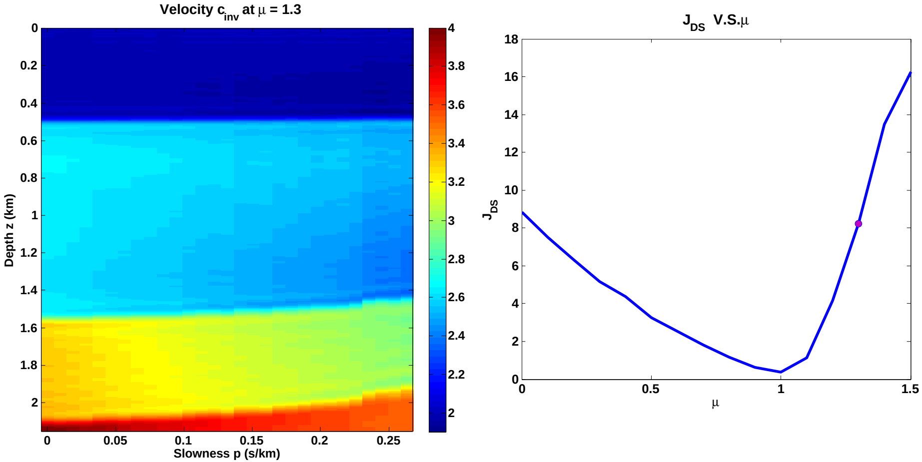
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \quad \mu = 1.2$$





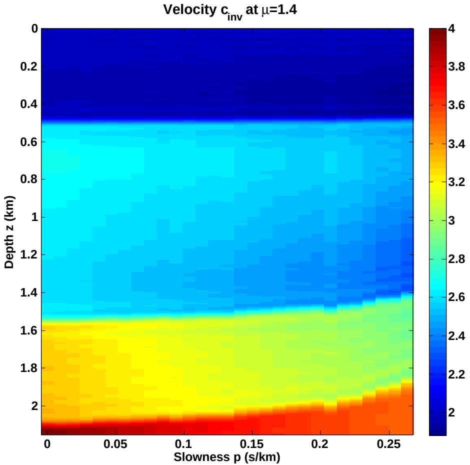
Scan Tests (Four-layer Model): $\mu = 1.3$

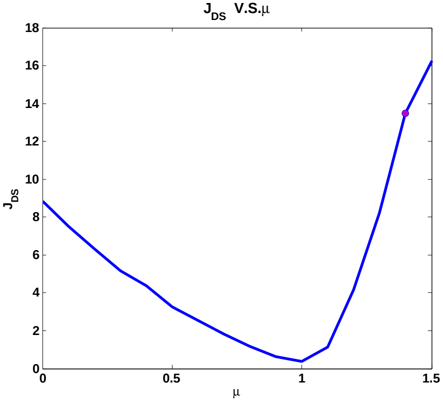
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \;\; \mu = 1.3$$



Scan Tests (Four-layer Model): $\mu = 1.4$

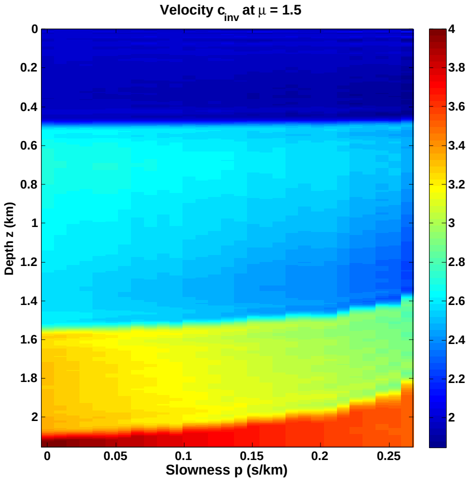
$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \;\; \mu = 1.4$$

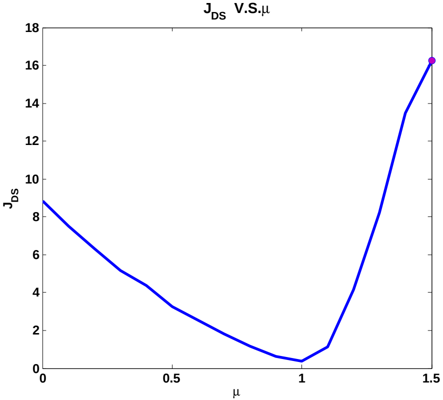




Scan Tests (Four-layer Model): $\mu = 1.5$

$$d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad \text{at} \;\; \mu = 1.5$$

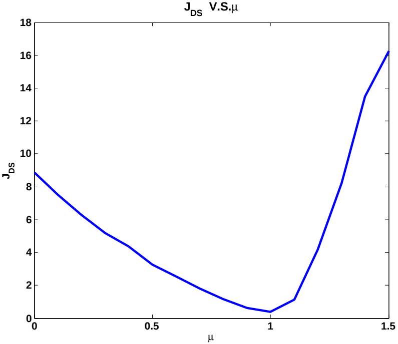


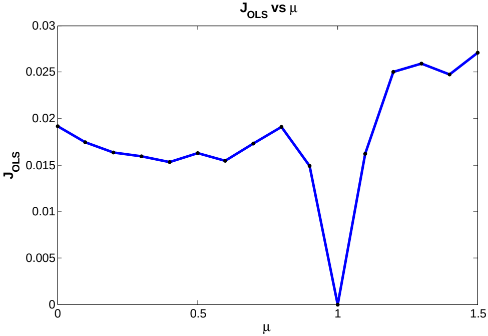


Scan J_{DSO} V.S. Scan J_{OLS}

Scan J_{OLS} along: $c_\mu(z) = (1-\mu)\, c_{hom} + \mu\, c^*(z) \quad (\mu \in [0, 1.5])$

Scan J_{DSO} along : $\quad d_\mu = (1-\mu)\, d_{lpert} + \mu\, d^* \quad (\mu \in [0, 1.5])$



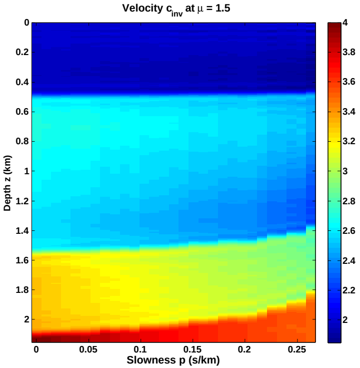


The DS Objective is :

- convex
- continuously differentiable $\cdots$ (Dong's MS thesis)

Improvement to the nonlinear DS Formulation

Recall:



1. Solutions to 1D subproblems achieve different accuracy, and the corresponding noise rapidly changes w.r.t. slowness
   $\longrightarrow$ increase of noise in DS objective
2. this approach cannot be extended to more general model
   e.g., general acoustic, multi-parameter inversion, ...

$\Longrightarrow$ Have to replace the 1D least-squares sub-inversions with one 2D least-squares problem with specific constraints to penalize the inconsistency in slowness direction

Outline

1. Overview of Waveform Inversion
2. Nonlinear DSO for plane-waves in layered media
3. Reformulation & Gradient Computation
4. Summary & Future Work

Acoustic System & Plane-wave Decomposition

Layered Acoustic Model

$$\begin{aligned}\frac{\partial p}{\partial t} + \kappa \nabla \cdot \mathbf{v} &= \omega(t)\delta(x,z) \\ \rho\frac{\partial \mathbf{v}}{\partial t} + \nabla p &= 0\end{aligned}$$

$\rho(z)$: density $\quad$ $p(x,z,t)$: pressure

$\kappa(z)$: bulk modulus $\quad$ $\mathbf{v}(x,z,t)$: particle velocity

$m(z) := (\rho(z), \kappa(z))$: model point $\quad$ $c(z) = \sqrt{\frac{\kappa}{\rho}}$: acoustic velocity

Introduce Radon transformed field

$$\begin{aligned}P(z,\xi,t) &= \int \mathrm{d}x \; p(x,z,t+\xi x), \\ \mathbf{V}(z,\xi,t) &= \int \mathrm{d}x \; \mathbf{v}(x,z,t+\xi x).\end{aligned}$$

Forward Map & Linearization

Radon Transform $\Longrightarrow$ a set of 1-D plane-wave problems

$$\begin{aligned} \left(1 - \frac{\kappa\xi^2}{\rho}\right)\frac{\partial P}{\partial t} + \kappa\frac{\partial \mathbf{V}_z}{\partial z} &= \omega(t)\delta\left(z\right) \\ \rho\frac{\partial \mathbf{V}_z}{\partial t} + \frac{\partial P}{\partial z} &= 0 \end{aligned}$$

Forward Map: $\mathcal{F}[m] := P(0, \xi, t)$

Extended Model Space: $\overline{\mathcal{M}} := \{\bar{m}(\xi, z) = (\bar{\rho}(\xi, z), \bar{\kappa}(\xi, z)) : \ldots\}$

Extended Modeling Operator: $\overline{\mathcal{F}} : \overline{\mathcal{M}} \longrightarrow \mathcal{D}$ defined by

the same equation system with m replaced by $\bar{m}$

Detailed derivation in Dong's technical report

Replace 1-D LS sub-problems with one 2-D LS problem with DS constraint

Differential Semblance Optimization problem:

$$\begin{aligned} \min_{d_l \in \mathcal{D}_l} \quad & J[d_l] := \tfrac{1}{2} \left\| \frac{\partial \bar{m}[d_l]}{\partial \xi} \right\|_W^2 \\ s.t. \quad & \bar{m}[d_l] = \underset{\bar{m} \in \overline{\mathcal{M}}}{\operatorname{argmin}} \, Q[\bar{m}], \end{aligned}$$

where

$$Q[\bar{m}] := \tfrac{1}{2} \left\{ \left\| \overline{\mathcal{F}}[\bar{m}] - (d_o + d_l) \right\|_{\mathcal{D}}^2 + \underbrace{\sigma^2 \left\| \frac{\partial \bar{m}}{\partial \xi} \right\|_W^2}_{\texttt{tie 1D-invs together}} \right\} + \mathcal{R}(\bar{m})$$

Predicted Advantages over the previous approach:

- Generalizability (constraint $\min_{\bar{m}} Q[\bar{m}]$ remains the similar form for general case)
- Better Numerical Performance (no evidence yet)

Sketch of Nonlinear DS Algorithm

Nonlinear DS Algorithm:

Initialization: set m^0, d_l^0, ω, ϵ, etc.

For $k = 0, 1, 2, \ldots$

1. Compute the sub-minimization problem to get
$$\bar{m}[d_l^k] = \underset{\bar{m}}{\operatorname{argmin}}\, Q[\bar{m}]$$
2. Compute $J^k = \frac{1}{2} \left\| \frac{\partial \bar{m}[d_l^k]}{\partial \xi} \right\|^2$. If $J^k \leq \epsilon J^0$, stop; else, continue
3. Compute $\nabla J[d_l^k]$. If $\left\| \nabla J[d_l^k] \right\| \leq \epsilon \left\| \nabla J[d_l^0] \right\|$, stop; else, continue
4. Compute d_l^{k+1} via descent method

Gradient Computation: Formula

Gradient Formula (detailed derivation in Dong's technical report)

$$\nabla J = -\Pi \; D\overline{\mathcal{F}}[\bar{m}] \; H_Q^{-1} \; \frac{\partial^2 \bar{m}}{\partial \xi^2}$$

where $H_Q = D\overline{\mathcal{F}}[\bar{m}]^T \, D\overline{\mathcal{F}}[\bar{m}] - \sigma^2 \frac{\partial^2}{\partial \xi^2} + D^2\mathcal{R}$

$\Pi : \mathcal{D} \longrightarrow \mathcal{D}_l$: Projector from data space onto low-frequency data space

$D^2\mathcal{R}$: the second derivative of $\mathcal{R}(\bar{m})$ with respect to $\bar{m}$

(in some case just constant, e.g., $\mathcal{R}(\bar{m}) = \gamma^2 \|\bar{m}\|^2 \Rightarrow D^2\mathcal{R} = 2\gamma^2$)

Gradient Computation: Explanation

Gradient Formula (detailed derivation in Dong's technical report)

$$\nabla J = -\Pi \underbrace{\overbrace{D\overline{\mathcal{F}}[\bar{m}] \underbrace{H_Q^{-1} \frac{\partial^2 \bar{m}}{\partial \xi^2}}_{\in \overline{\mathcal{M}}}}^{\in \mathcal{D}}}_{\in \mathcal{D}_l}$$

where $H_Q = D\overline{\mathcal{F}}[\bar{m}]^T \, D\overline{\mathcal{F}}[\bar{m}] - \sigma^2 \frac{\partial^2}{\partial \xi^2} + D^2\mathcal{R}$

$\Pi : \mathcal{D} \longrightarrow \mathcal{D}_l$: Projector from data space onto low-frequency data space

H_Q: Gauss-Newton Hessian for $Q[\bar{m}]$ with

$$H_Q \; \delta\bar{m} = \underbrace{D\overline{\mathcal{F}}[\bar{m}]^T \; \delta d}_{\in \overline{\mathcal{M}}} = D\overline{\mathcal{F}}[\bar{m}]^T \; \Pi^T \; \delta d_l$$

Gradient Computation: Procedure

Gradient Computation:

Given $\bar{m}[d_l] = \mathrm{argmin}_{\bar{m}}\, Q[\bar{m}]$, need to

1. compute $\mathbf{b} \approx -\frac{\partial^2 \bar{m}}{\partial \xi^2}$
2. solve $H_Q \mathbf{q} = \mathbf{b}$ for $\mathbf{q}$ via CG algorithm

   need to compute the action of H_Q on any vector $\mathbf{g}$,

   i.e., compute $\left(D\overline{\mathcal{F}}[\bar{m}]^T \, D\overline{\mathcal{F}}[\bar{m}] - \sigma^2 \frac{\partial^2}{\partial \xi^2} + D^2\mathcal{R} \right) \mathbf{g}$:

   - $D^2\mathcal{R}\ \mathbf{g}$ easy to compute
   - $\frac{\partial^2}{\partial \xi^2}\ \mathbf{g}$ easy to compute
   - $\mathbf{w} = D\overline{\mathcal{F}}[\bar{m}]\ \mathbf{g}$ via one forward propagation

     & $D\overline{\mathcal{F}}[\bar{m}]^T\ \mathbf{w}$ via adjoint state computation
3. compute $\nabla J \approx \Pi\ D\overline{\mathcal{F}}[\bar{m}]\ \mathbf{q}$ via one forward propagation and one projection (filter)

Outline

Summary & Future Work

Done:

- Formulated WI via Extended Modeling (in Appendix)
- Proposed a nonlinear DS strategy for the simplest model
- Illustrated the properties of the proposed DS objective via scan tests
- Introduced one important revision to the proposed DS approach
- Derived gradient computation

Plan:

- Implement gradient computation
- Implement inversion for general acoustic model

  (based on existing packages SEAMX, TSOpt, ...)
- Consider further extension
- ...

Thank You!

Appendix

Gradient Computation: derivation

- $J[d_l] = \frac{1}{2} \left\| \frac{\partial \bar{m}[d_l]}{\partial \xi} \right\|^2_{\mathsf{M}} \Rightarrow \delta J = - \left(\frac{\partial^2 \bar{m}}{\partial \xi^2}, \delta \bar{m} \right)_{\mathsf{M}}$
- First-order necessity condition of the sub-minimization problem

$$\nabla_{\bar{m}} Q = 0$$

where

$$\nabla_{\bar{m}} Q = D\overline{\mathcal{F}}[\bar{m}]^T \left(\overline{\mathcal{F}}[\bar{m}] - (d_o + d_l) \right) - \sigma^2 \frac{\partial}{\partial \xi} \triangle^{-1} \frac{\partial}{\partial \xi} \bar{m} + D\mathcal{R}(\bar{m})$$

$\Rightarrow$

$$H_Q \, \delta \bar{m} = D\overline{\mathcal{F}}[\bar{m}]^T \delta d_l$$

where

$$H_Q := D\overline{\mathcal{F}}[\bar{m}]^T D\overline{\mathcal{F}}[\bar{m}] - \sigma^2 \frac{\partial}{\partial \xi} \triangle^{-1} \frac{\partial}{\partial \xi} + D^2 \mathcal{R}$$

$\Rightarrow$

$$\delta \bar{m} = D_{d_l} \bar{m} \; \delta d_l$$

$$D_{d_l} \bar{m} = H_Q^{-1} D\overline{\mathcal{F}}[\bar{m}]^T$$

Gradient Computation: derivation

$$\begin{aligned} \delta J &= -\left(\frac{\partial^2 \bar{m}}{\partial \xi^2}, \delta \bar{m}\right)_{\overline{\mathcal{M}}} \\ &= -\left(\frac{\partial^2 \bar{m}}{\partial \xi^2}, D_{d_l} \bar{m}\ \delta d_l\right)_{\overline{\mathcal{M}}} \\ &= -\left((D_{d_l}\ \bar{m})^T\, \frac{\partial^2 \bar{m}}{\partial \xi^2}, \delta d_l\right)_{\mathcal{D}} \end{aligned}$$

Hence,

$$\begin{aligned} \nabla J &= -\,(D_{d_l}\ \bar{m})^T\ \tfrac{\partial^2 \bar{m}}{\partial \xi^2} \\ &= -D\overline{\mathcal{F}}[\bar{m}]\ H_Q^{-1}\ \tfrac{\partial^2 \bar{m}}{\partial \xi^2} \end{aligned}$$

Recall

$$H_Q = D\overline{\mathcal{F}}[\bar{m}]^T D\overline{\mathcal{F}}[\bar{m}] - \sigma^2 \frac{\partial}{\partial \xi} \triangle^{-1} \frac{\partial}{\partial \xi} + D^2 \mathcal{R}$$

Key computation

$$H_Q\ q = b$$

such as

$$H_Q \delta \bar{m} = -\nabla_{\bar{m}} Q \quad , \quad H_Q \delta \bar{m} = D\overline{\mathcal{F}}[\bar{m}]^T \delta d_l$$

Formulate WI via Extended Modeling

Extended Modeling Concept $\longrightarrow$ a unified view of OLS and MVA (Symes, 2008)

The *extension* of model $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{D}$ consists of

- $\overline{\mathcal{M}}$: extended model space
- $E : \mathcal{M} \longrightarrow \overline{\mathcal{M}}$: extension operator, one-to-one, $E[\mathcal{M}] \subset \overline{\mathcal{M}}$ ($E[\mathcal{M}]$: the "physical models")
- $\overline{\mathcal{F}} : \overline{\mathcal{M}} \longrightarrow \mathcal{D}$: extended modeling operator, $\mathcal{F}[m] = \overline{\mathcal{F}}[E[m]]$ for any $m \in \mathcal{M}$

Extended inversion:

given $d \in \mathbf{D}$, find $\bar{m} \in \overline{\mathcal{M}}$ such that $\overline{\mathcal{F}}[\bar{m}] \simeq d$

solution $\bar{m}$ physically meaningful only if $\bar{m} \in E[\mathcal{M}]$

Since $\overline{\mathcal{M}}$ has more degrees of freedom, ambiguity is more likely.

Formulate WI via Extended Modeling: Annihilator

Inverse problem: look for $\bar{m}$ so that

(1) $\bar{m} \in E[\mathcal{M}]$, i.e., $\bar{m} = E[m]$ for some $m \in \mathcal{M}$

(2) $\overline{\mathcal{F}}[\bar{m}] \simeq d$

Then, m is the solution to the original problem.

Need to seek objectives whose extrema represent the solution

Define *annihilator* A: $\overline{\mathcal{M}} \longrightarrow \mathcal{H}$ so that

$$\bar{m} \in E[\mathcal{M}] \Longleftrightarrow A\,\bar{m} = 0.$$

A general form of the inverse problem

$$\begin{array}{ll} \min_{\bar{m} \in \overline{\mathcal{M}}} & J_A[\bar{m}] := \frac{1}{2}\,\|A\,\bar{m}\|^2_{\mathcal{H}} \\ s.t. & \|\overline{\mathcal{F}}[\bar{m}] - d\|^2_{\mathcal{D}} \simeq 0 \end{array}$$

Question: why consider this problem instead of traditional OLS problem?

Extended Modeling may lead to Effective WI

Extension concept (Symes,2008)

- provides a unified view of WI and MVA
  in linearized extended modeling context, MVA is a solution method to the partially linearized inverse problem

- has lots of familiar extensions
  annihilator A chosen in differential semblance class , lots of successful implementations and theoretical results
  (Symes(1990), Symes & Carazzone(1991), Symes(1999),
  Shen & Calandra(2005),...)
- suggests an approach to nonlinear waveform inversion incorporating elements of MVA
  Symes(1991) proved this problem is equivalent to an unconstrained problem with no local minima and the objective has stable shape independent of source spectrum (under some assumption ...)