Mass Lumping for Constant Density Acoustics and Accuracy at Interfaces

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Discretization Approaches

Finite Differences:

+ Easy to implement
+ Reasonable balance of accuracy and efficiency
− Loss of accuracy in presence of discontinuities
  related concepts: stairstep diffraction, interface misalignment
− Complex geometry (e.g. surface relief)
− Adaptive grids

Finite Elements:

+ Higher order in presence of discontinuities
+ Complex geometry
+ Unstructured grids
− **Linear system at each time step (conforming FEs)**

Finite Elements Method

Acoustic wave equation:

$$\frac{1}{c^2(x)} p_{tt}(x,t) - \nabla^2 p(x,t) = 0$$

Weak formulation:

$$\int_\Omega \frac{p_{tt}(x,t) v(x)}{c^2(x)} + \int_\Omega \nabla p(x,t) \cdot \nabla v(x) = 0, \quad \forall v \in V$$

Discretization: $V = V_h; \quad p(x,t) = \sum_j \hat{p}_j(t)\, v_j(x), \; v_j \in V$

$$\mathbf{M}\hat{\mathbf{p}}_{\mathbf{tt}} + \mathbf{S}\hat{\mathbf{p}} = \mathbf{0}, \quad \hat{p} = [\hat{p}_1, \hat{p}_2, \ldots]^{\mathsf{T}}$$

$$m_{ij} = \int_\Omega \frac{v_i(x)\, v_j(x)}{c^2(x)}, \quad s_{ij} = \int_\Omega \nabla v_i(x) \cdot \nabla v_j(x)$$

RICE

Mass Lumping

Mass lumping replaces mass matrix M with a diagonal one

Well-known as $\quad d_i = \sum_j m_{ij} \quad$ (P^1 or Q^1 elements)

Heuristic justification:

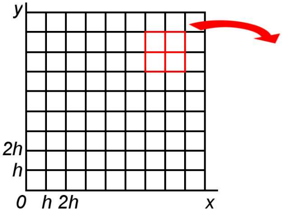
$$\sum_{j \in N(i)} m_{ij} \frac{d^2 \hat{p}_j}{dt^2} = \text{r.h.s}$$

Sufficiently fine mesh: $\hat{p}_j \approx \hat{p}_i,\ j \in N(i)$
Therefore:

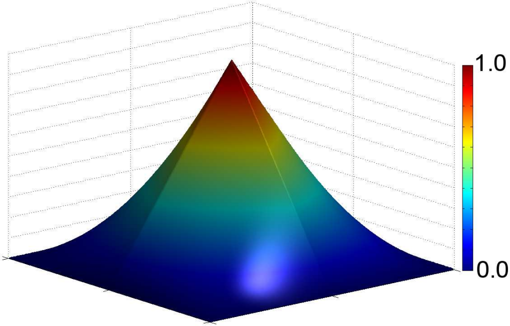
$$\sum_{j \in N(i)} m_{ij} \frac{d^2 \hat{p}_j}{dt^2} \approx \left(\sum_{j \in N(i)} m_{ij} \right) \frac{d^2 \hat{p}_i}{dt^2}$$

Q^1 Elements

Q^1 base functions – tensor products of 1D “hat” functions



2D base function:



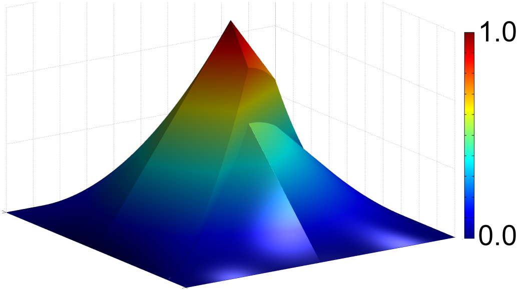
FEM $\sim$ FD with Averaging

2D FEM $\rightsquigarrow$ FD methods (9/5-point stencil), **but ...**

Variable stencil coefficients:

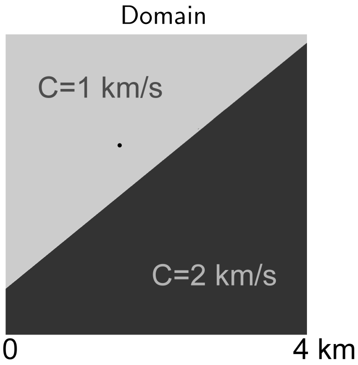
$$m_{ij} = \int_\Omega \frac{v_i(x)\, v_j(x)}{c^2(x)}, \quad \textbf{lumped: } d_i = \sum_j m_{ij} = \int_\Omega \frac{v_i(x)}{c^2(x)}$$

$v_i(x)/c^2(x)$:



Numerical Experiment

- Domain: 4 km × 4 km; dipping interface
- Simulation time: 0.5 s
- Source: Ricker, 15 Hz
- Discretization: 2-2 (5-point stencil), 1200 × 1200 grid points



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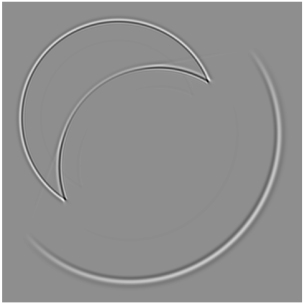
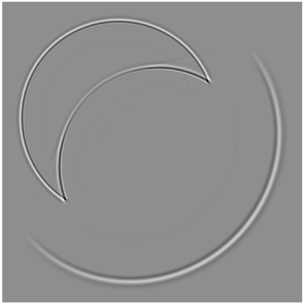
Q^1 mass-lumped FE



0 4 km

2-2 FD



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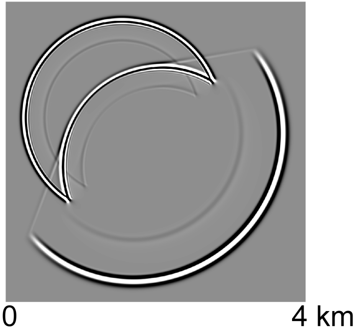
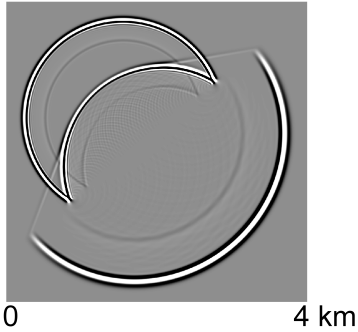


2-2 FD



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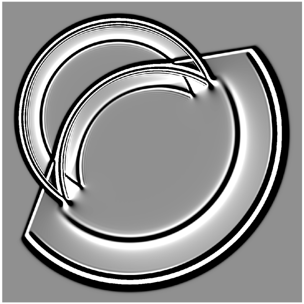

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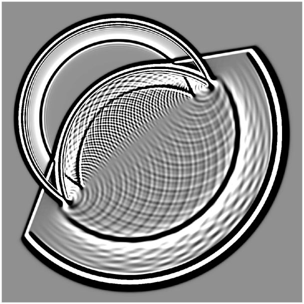

0 4 km

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Mass Lumping – Theory

Quadrature rules with points coinciding with nodes $\rightsquigarrow$
diagonal mass matrix

Justification uses quadrature error estimate and relies on the regularity of the coefficients

Theoretical Result (Symes):

- Constant density acoustic wave equation
- Solutions *smooth in time*
- L^∞ coefficients

$\Rightarrow$ Mass-lumped approximation with Q^1 elements *preserves the convergence order*

Future Work

Theoretical justification for:

- Variable density acoustics
- First order systems (via mixed FEs) $\rightsquigarrow$ elastics

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THANK YOU !