Operator-Based Upscaling for the Elastic Wave Equation

Tetyana Vdovina
Susan Minkoff, Sean Griffith

The Rice Inversion Project
vdovina@rice.edu

February 20, 2009

Operator-Based Upscaling

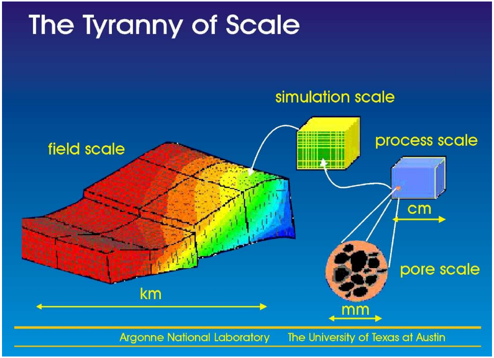
- Upscaling in the Context of Multiscale Methods
- Upscaling for the Elastic Wave Equation
  - Description of the Method
  - Numerical Implementation
  - Numerical Experiment
- Current and Future Work

Multiscale Methods

- Why do we need multiscale methods?
  - Many processes in nature involve multiple scales.
- **Goal:** to design a numerical technique that
  - produces accurate solution on the coarse scale;
  - is more efficient than solving full fine scale problem.

Multiscale problems:

- flow in porous media (10^{-2} - 10^{4} m),
- composite materials (10^{-9} m - large scales depend on applications),
- protein folding (10^{-15} - 10^{-1} s).



http://www.ticam.utexas.edu/Groups/SubSurfMod/ACTI/IPARS.htm

Upscaling Methods

Highly detailed physical models $\xrightarrow{\text{Upscaling}}$ Feasible simulation grids

Upscaling is the process of converting the problem from the fine scale where physical parameters are defined to a coarse scale.

- Averaging: Review by Renard and Marsily (1997).
- Renormalization: King (1989).
- Homogenization: Bensoussan, Lions, Papanicolaou (1978).
- Multiscale FEM: Hou, Wu (1997).
- Mortar Upscaling: Peszynska, Wheeler, Yotov (2002).
- Variational Multiscale Method: Hughes (1995).
- Operator Upscaling: Arbogast, Minkoff, Keenan (1998).
- Metric Upscaling: Owhadi et al. (2006).

Numerical Simulation of Seismograms

SEG Advanced Modeling "SEAM" project, Phase 1 model:

- simulates typical deep water sub-salt exploration regime,
- 28 km (W-E) $\times$ 30 km (N-S) $\times$ 15 km (depth) and 15 s.

Computational cost:

- 10 m grid $\implies$ 10^{10} spatial grid points,
- source bandwidth: 0 - 30 Hz $\implies$ 30000 time steps,
- number of simulations: 100000.

Acoustic: 20 FLOP per point $\implies$ 10^{20} FLOP:

3000 years on a 1 GFLOPS desktop

Elastic: 100 FLOP per point $\implies$ $5 \cdot 10^{20}$ FLOP:

15000 years on a 1 GFLOPS desktop

Model problem: The Elastic Wave Equation

- Velocity/displacement formulation of the elastic equation:

$$\rho(\mathbf{x})\frac{\partial \mathbf{v}(t,\mathbf{x})}{\partial t} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f},$$
$$\rho(\mathbf{x})\frac{\partial \mathbf{u}(t,\mathbf{x})}{\partial t} = \rho(\mathbf{x})\mathbf{v}(t,\mathbf{x}),$$

  ρ is density, $\boldsymbol{\sigma}$ is the stress tensor.

- Weak formulation, Komatitsch et al. (1999):

$$\begin{aligned} \left(\rho\frac{\partial \mathbf{v}}{\partial t}, \mathbf{w}\right) &= -\left(\boldsymbol{\sigma}, \nabla \mathbf{w}\right) + \left(\mathbf{f}, \mathbf{w}\right), \\ \left(\rho\frac{\partial \mathbf{u}}{\partial t}, \mathbf{w}\right) &= \left(\rho \mathbf{v}, \mathbf{w}\right). \end{aligned}$$

- Eliminate components of the stress tensor:

$$\sigma_{i,j} = \lambda \sum_{k}^{3} \frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Weak Formulation (continued)

- First component of velocity:

$$\begin{aligned}\left(\rho\frac{\partial v_1}{\partial t}, w\right) &= -\left((\lambda+2\mu)\frac{\partial u_1}{\partial x} + \lambda\frac{\partial u_2}{\partial y} + \lambda\frac{\partial u_3}{\partial z}, \frac{\partial w}{\partial x}\right) \\ &- \left(\mu\frac{\partial u_1}{\partial y} + \mu\frac{\partial u_2}{\partial x}, \frac{\partial w}{\partial y}\right) \\ &- \left(\mu\frac{\partial u_1}{\partial z} + \mu\frac{\partial u_3}{\partial x}, \frac{\partial w}{\partial z}\right) + (f_1, w)\,.\end{aligned}$$

- First component of displacement:

$$\left(\rho\frac{\partial u_1}{\partial t}, w\right) = (\rho v_1, w)\,.$$

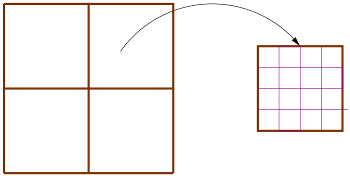
- Upscale both variables.

Two-Scale Decomposition

Goal: Capture fine-scale behavior on the coarse grid.

Idea: Use a two-scale decomposition of solutions.

- Two-scale grid:



- Two-scale decomposition:

$$\mathbf{v} = \mathbf{v}^c + \delta\mathbf{v},$$
$$\mathbf{u} = \mathbf{u}^c + \delta\mathbf{u},$$

  - $\mathbf{v}^c$, $\mathbf{u}^c$ are the coarse-scale unknowns,
  - $\delta\mathbf{v}$, $\delta\mathbf{u}$ are the subgrid unknowns internal to each block.

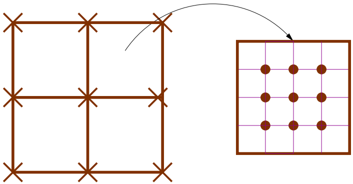
- Simplifying assumption: Subgrid solutions are equal to zero on coarse block boundaries.

Two-Scale Decomposition

Goal: Capture fine-scale behavior on the coarse grid.

Idea: Use a two-scale decomposition of solutions.

- Two-scale grid:



- Two-scale decomposition:

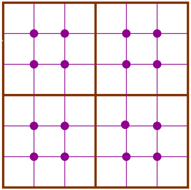
$$\mathbf{v} = \mathbf{v}^c + \delta\mathbf{v},$$
$$\mathbf{u} = \mathbf{u}^c + \delta\mathbf{u},$$

  - $\mathbf{v}^c$, $\mathbf{u}^c$ are the coarse-scale unknowns,
  - $\delta\mathbf{v}$, $\delta\mathbf{u}$ are the subgrid unknowns internal to each block.

- Simplifying assumption: Subgrid solutions are equal to zero on coarse block boundaries.

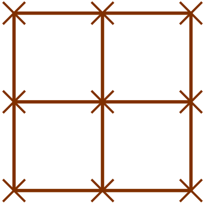
Two-stage Algorithm

Step 1: On each coarse element solve the subgrid problem:



- Zero boundary conditions
- Basis: piecewise trilinear functions
- Quadrature: trapezoid rule
- Mass matrix: diagonal

Step 2: Use the subgrid solutions to solve the coarse-grid problem:



- Original boundary conditions
- Original fine-scale parameter fields
- Basis: piecewise trilinear functions
- Quadrature: subgrid trapezoid rule
- Mass matrix: banded (27 diagonals) and sparse

Parallel Implementation

Preprocessing:

- read input data and split it among processes,
- construct coarse grid system matrix.

Time-step loop:

- Subgrid problems: embarrassingly parallel
  - no communication between processors,
  - no additional ghost-cell memory allocations,
  - diagonal linear system.
- Coarse problem:
  - Construct rhs locally and assemble global copy on all processes:
    - 3 velocity load vectors, 10 inner products each,
    - 3 displacement load vectors, 3 inner products each,
    - example: $20 \times 20 \times 20$ coarse grid blocks
      $20^3 \cdot (3 \cdot 10 + 3 \cdot 3) = 169,744$ triple integrals.
  - Solve linear system using `SuperLU_DIST` or `UMFPACK`.

Postprocessing: reconstruct $\mathbf{v} = \mathbf{v}^c + \delta\mathbf{v}$, $\mathbf{u} = \mathbf{u}^c + \delta\mathbf{u}$.

Parallel Performance

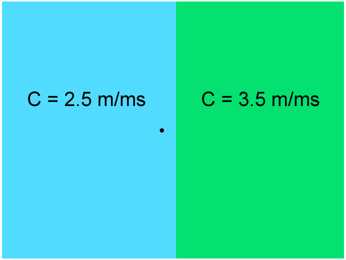
Number of processes	Time-step loop	Subgrid problems	Coarse problem
1	238.07	17.00	220.23
2	119.41	8.39	110.63
4	60.03	4.22	55.62
8	30.61	2.23	28.28
16	17.07	1.00	16.02
32	8.17	0.51	7.63
64	4.51	0.25	4.24
128	2.88	0.13	2.74

- Discretization: $320 \times 320 \times 320$ fine grid blocks, $32 \times 32 \times 32$ coarse grid blocks, 20 time steps.
- Full finite element code on a single process: 140.30 seconds.

Acoustic Numerical Example

- Pressure-acceleration formulation:

$$\mathbf{u}(x,z,t) = -\nabla p(x,z,t),$$
$$\frac{1}{c^2(x,z)}\frac{\partial^2 p(x,z,t)}{\partial t^2} + \nabla\cdot\mathbf{u}(x,z,t) = w(t)\delta(x,z)$$



- Upscale acceleration only
- Domain: 1000×1000 m and 250 ms
- Source: Ricker wavelet, peak frequency 15 Hz
- Fine grid: 200×200, coarse grid: 20×20

Acoustic Numerical Example: Pressure

play

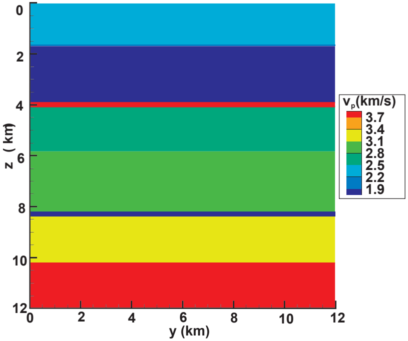
Acoustic Numerical Example: Horizontal Acceleration

play

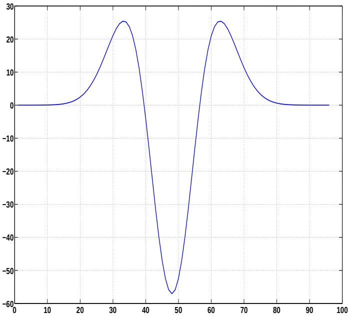
Numerical Experiment I

- Domain: $12 \times 12 \times 12$ km and 1.56 seconds.
- Source: $\mathbf{f}(t, \mathbf{x}) = Ah(t)g(|\mathbf{x} - \mathbf{x}_s|^2)\mathbf{a}$,
  - $h(t)$ is Ricker wavelet with peak frequency 1.7 Hz,
  - $g(|\mathbf{x} - \mathbf{x}_s|^2)$ is Gaussian.
- Fine grid: $120 \times 120 \times 120$, coarse grid $24 \times 24 \times 24$.
- Layered medium.

Compressional Velocity

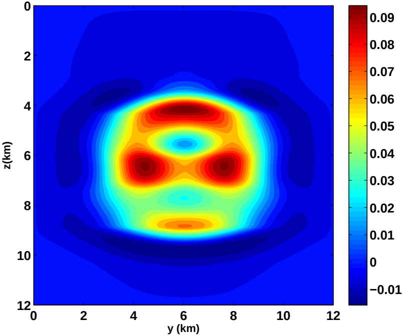


Ricker Source

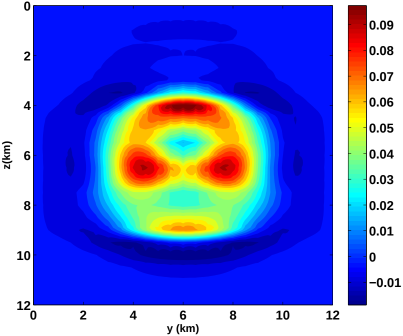


1st Component of the Velocity Solution (*yz*-plane), 3.7 km

Full finite-element solution

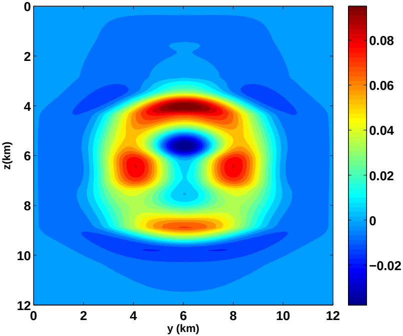


Reconstructed upscaled solution $v_1^c + \delta v_1$

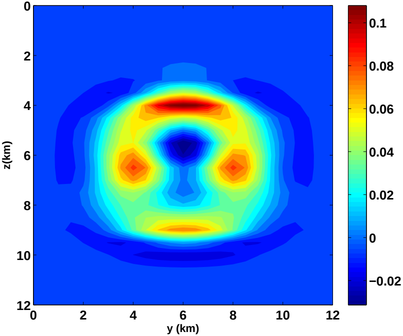


1st Component of the Velocity Solution (yz-plane), 4.0 km

Full finite-element solution



Coarse solution v_1^c



Summary

What do we have?

- Elastic wave equation:
  - velocity/displacement formulation, 3D,
  - serial and parallel implementations,
  - numerical convergence.
  - Vdovina, Griffith, Minkoff (in revision)
- Acoustic wave equation:
  - pressure/acceleration formulation, 2D,
  - serial and parallel implementations,
  - convergence analysis confirmed by numerical experiments.
  - Vdovina, Minkoff, Korostyshevskaya (2005), Korostyshevskaya, Minkoff (2006), Vdovina, Minkoff (2008)

Where do we go with this?

- Seismic inversion: progress report in my second talk

Numerical Convergence

- Homogeneous medium
- Source function is chosen to produce closed form solutions
- Both fine and coarse grids are refined

Number of fine blocks	Number of coarse blocks	Number of time steps	$\frac{\|\|V_1 - v_1\|\|_\infty}{\|\|V_1\|\|_\infty}$	Rate
$50 \times 50 \times 50$	$5 \times 5 \times 5$	50	2.4554e-01	–
$100 \times 100 \times 100$	$10 \times 10 \times 10$	100	5.5976e-02	2.1
$200 \times 200 \times 200$	$20 \times 20 \times 20$	200	1.5578e-02	1.9
$400 \times 400 \times 400$	$40 \times 40 \times 40$	400	3.8776e-03	2.0

Numerical Convergence (cont.)

- Fine grid is fixed, coarse grid is refined

Number of fine blocks	Number of coarse blocks	Number of time steps	$\frac{\|\|V_1 - v_1\|\|_\infty}{\|\|V_1\|\|_\infty}$	Rate
$200 \times 200 \times 200$	$5 \times 5 \times 5$	200	2.4770e-01	–
$200 \times 200 \times 200$	$10 \times 10 \times 20$	200	5.6439e-02	2.1
$200 \times 200 \times 200$	$20 \times 20 \times 20$	200	1.5578e-02	1.9
$200 \times 200 \times 200$	$40 \times 40 \times 40$	200	3.7272e-03	2.1

- Fine grid is refined, coarse grid is fixed

Number of fine blocks	Number of coarse blocks	Number of time steps	$\frac{\|\|V_1 - v_1\|\|_\infty}{\|\|V_1\|\|_\infty}$	Rate
$50 \times 50 \times 50$	$10 \times 10 \times 10$	50	5.5967e-02	–
$100 \times 100 \times 100$	$10 \times 10 \times 10$	100	5.7067e-02	–
$200 \times 200 \times 200$	$10 \times 10 \times 10$	200	5.6439e-02	–
$400 \times 400 \times 400$	$10 \times 10 \times 10$	400	5.7506e-02	–